Chapter 5

Integer Linear Programming

Chapter Objectives

- All Integer, Mixed Integers, Binary Models.
- Graphical representation.
- Effects of Rounding.
- Solution approaches.
- Computer solution to Integers Models.
- Lack of Sensitivity Analyses.
- Use of Binary Variables.
  - Capital Budgeting / Goal Accomplishment constraints.
  - K out of N Constraints *must hold*..
  - Fixed Charge, Facility Location Models.

5.1 Introduction

- Many times, some or all of the decision variables must be restricted to integer values.
- Examples:
  - The number of aircraft purchased this year.
  - The number of machines needed for production.
  - The number of trips made by a sales person.
  - The number of police officers assigned to the nightshift

- Integer variables may be required when the model represents a one time decision (not an ongoing operation).

- Integer Linear Programming (ILP) models are much more difficult to solve than Linear Programming (LP) models.

- Algorithms that solve integer linear models do not provide valuable sensitivity analysis results.

- ILP models can be classified as follows:
  - ***All integer*** - all the variables are integers.
  - ***Mixed integer*** - some variables, but not all, are integers.
  - ***Binary*** - all the variables are either 0 or 1.

5.2 Complexities of ILPS

- If an integer model is solved as a simple linear model, at the optimal solution non-integer values may be attained.

- Rounding to integer values may result in:
  - Infeasible solutions
  - Feasible but not optimal solutions
  - Optimal solutions.

- Why not enumerate all the feasible integer points and select the best one?
  - Enumerating all the integer solutions is impractical because of the large number of feasible integer points.

- Is rounding ever done? Yes, particularly if -
  - The values of the positive decision variables are relatively large, and
  - The values of the objective function coefficients are relatively small.

The following example illustrates some of the complications arise when integer restrictions are placed on decision variables.

BOXCAR BURGER RESTAURANTS

- Boxcar Burger is a new chain of fast-food establishments.
- Boxcar is planning expansion in the downtown and suburban areas.
- Management would like to determine how many restaurants to open in each area in order to maximize net weekly profit.

- Requirements and restrictions
  - No more than 19 managers can be assigned.
  - At least two downtown restaurants are to be opened.
  - Total investment cannot exceed $2.7 Million.

- Data

	Suburban	Downtown
Investment per location	200,000	600,000
Daily profit	1,200	2,000
Operation hours	24 hours	12 hours
Number ofmanagers needed	3	1

SOLUTION

- Decision Variables

  X1 = Number of suburban boxcar burger restaurants to be opened.

  X2 = Number of downtown boxcar burger restaurants to be opened.

- The mathematical model is formulated next

Net weekly profit

Max 1200X1+ 2000X2

ST:

Total investment cannot exceed $2.7 dollars

$$2X1 + 6X2 \le 2.7$$

At least 2 downtown restaurants

$$X2 \ge 2$$

Not more than 19 managers can be assigned

$$3X1 + X2 \ge 19$$

X1, X2 are non-negative integers

- Constraints
  - The total investment cannot exceed 2.7$ million

5.3 Sensitivity in ILP

- In ILP models, there is no pattern to the disjoint effects of changes to the objective function and right hand side coefficients.
- When changes occur, they occur in big ”steps,” rather than the smooth, marginal fashion experienced in linear programming.
- Therefore, sensitivity analysis for integer models must be made by re-solving the problem, a very time-consuming process.

5.4 Mixed Integer Linear Programming

- A mixed integer linear programming model is one in which some, but not all, the variables are restricted integers.

- The Shelly Mednick Investment Problem illustrates this situation

SHELLY MEDNICK INVESTMENT PROBLEM

- Shelley Mednick has decided to give the stock market a try.
- She will invest in
  - TCS, a communication company stock, and or,
  - MFI, a mutual fund.
- Shelley is a cautious investor. She sets limits on the level of investments, and a modest goal for gain for the year.

- Data
  - TCS is been sold now for $55 a share.
  - TCS is projected to sell for $68 a share in a year.
  - MFI is predicted to yield 9% annual return.

- Restrictions
  - Expected return should be at least $250.
  - The maximum amount invested in TCS is not to exceed 40% of the total investment.
  - The maximum amount invested in TCS is not to exceed $750.

SOLUTION

- Decision variables
  - X1 = Number of shares of the TCS purchased.
  - X2 = Amount of money invested in MFI.
- The mathematical model

Minimize 55X1 + X2

ST

$$13X1 + 0.09X2 \geq 250$$

$$33X1 - 0.40X2 \leq 0$$

$$55X1 \leq 750$$

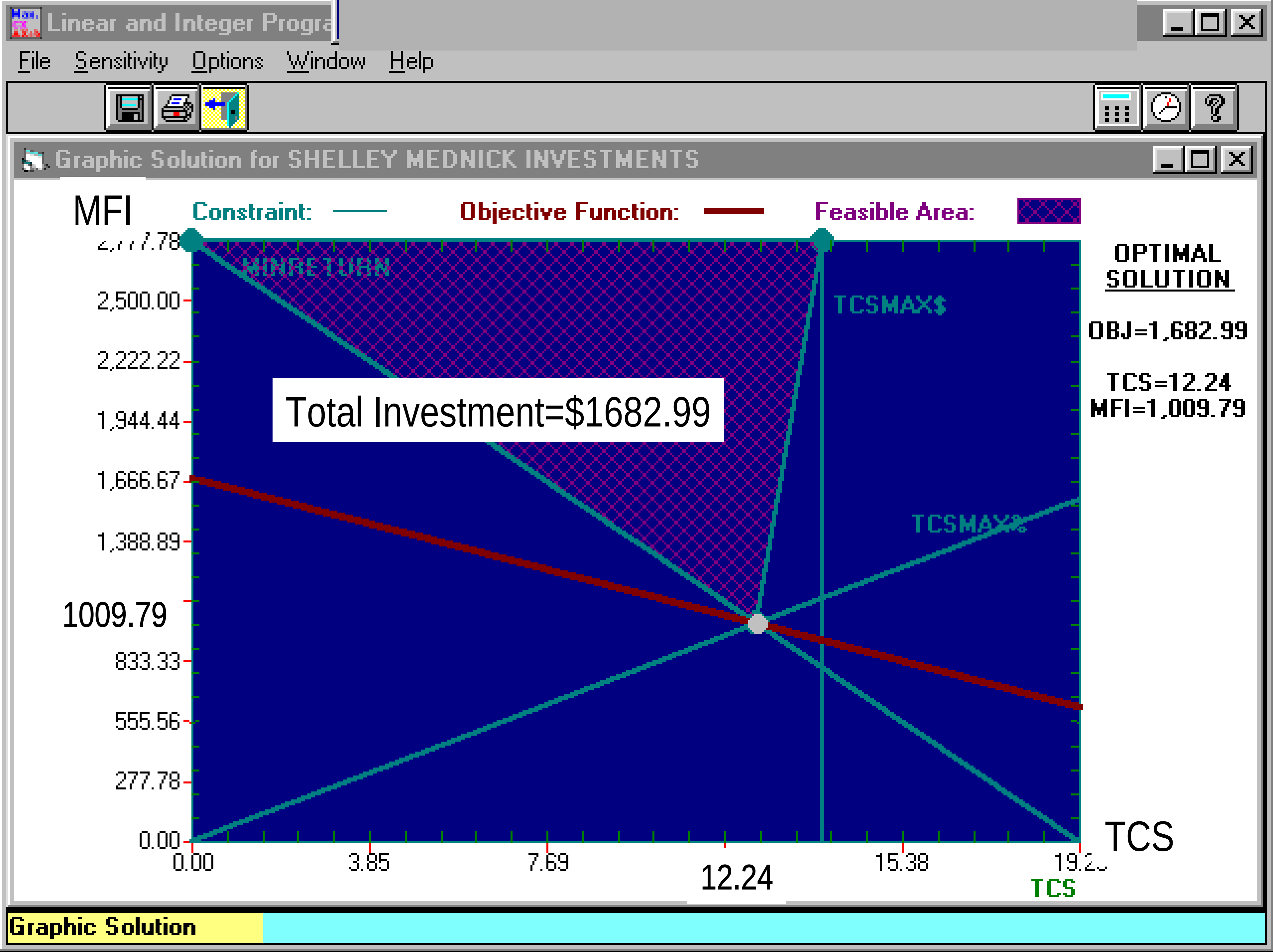
$$X1, X2 \geq 0$$

X1 integer.

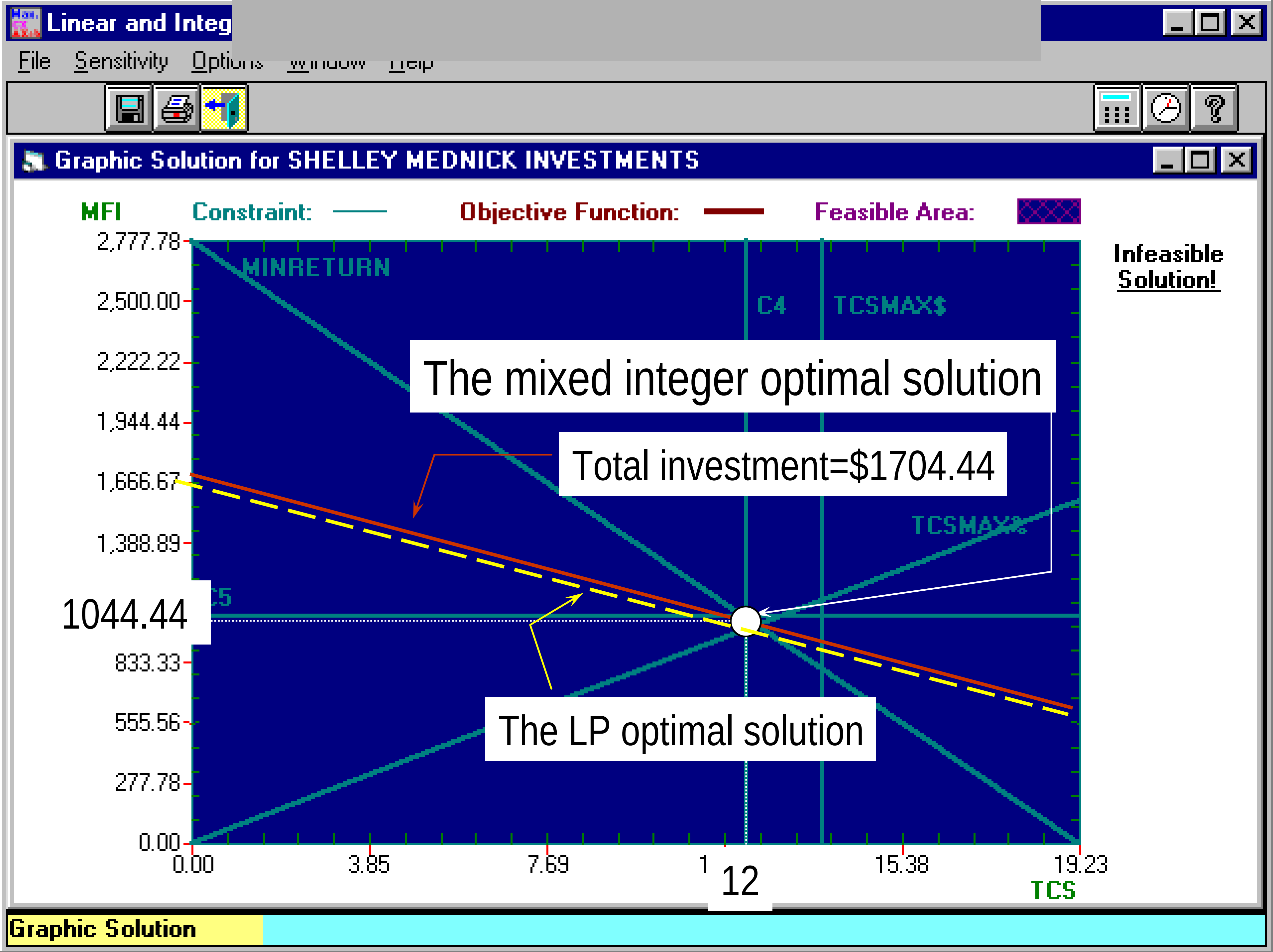
Projected yearly return

Not more than 40%

Not more than $750 in TCS



Linear and Integer Progra
File Sensitivity Options Window Help
Graphic Solution for SHELLEY MEDNICK INVESTMENTS
MFI
Constraint:
Objective Function:
Feasible Area:
MINRETURN
TCSMAX$
TCSMAX%
Total Investment=$1682.99
OPTIMAL SOLUTION
OBJ=1,682.99
TCS=12.24
MFI=1,009.79
2,500.00
2,222.22
1,944.44
1,666.67
1,388.89
1009.79
833.33
555.56
277.78
0.00
0.00
3.85
7.69
12.24
15.38
TCS
Graphic Solution


Linear and Integ
File Sensitivity Options Window Help
Graphic Solution for SHELLEY MEDNICK INVESTMENTS
MFI
Constraint:
Objective Function:
Feasible Area:
Infeasible Solution!
MINRETURN
C4
TCSMAX$
TCSMAX%
C5
The mixed integer optimal solution
Total investment=$1704.44
The LP optimal solution
1044.44
12
2,777.78
2,500.00
2,222.22
1,944.44
1,666.67
1,388.89
833.33
555.56
277.78
0.00
0.00
3.85
7.69
15.38
19.23
TCS
Graphic Solution

5.6 Personnel Scheduling Problem

Sunset Beach Lifeguard Assignments

- The City of Sunset Beach staffs lifeguards 7 days a week.
- Regulations require that city employees work five days.
- Insurance requirements mandate 1 lifeguard per 8000 average daily attendance on any given day.
- The city wants to employ as few lifeguards as possible.

SOLUTION

- Problem Summary
  - Schedule lifeguard over 5 consecutive days.
  - Minimize the total number of lifeguards.
  - Meet the minimum daily lifeguard requirements (see the linear model next).

- Data
  - For each day, at least the minimum required lifeguards must be on duty.

Sun.	Mon.	Tue.	Wed.	Thr.	Fri.	Sat.
8	6	5	4	6	7	9

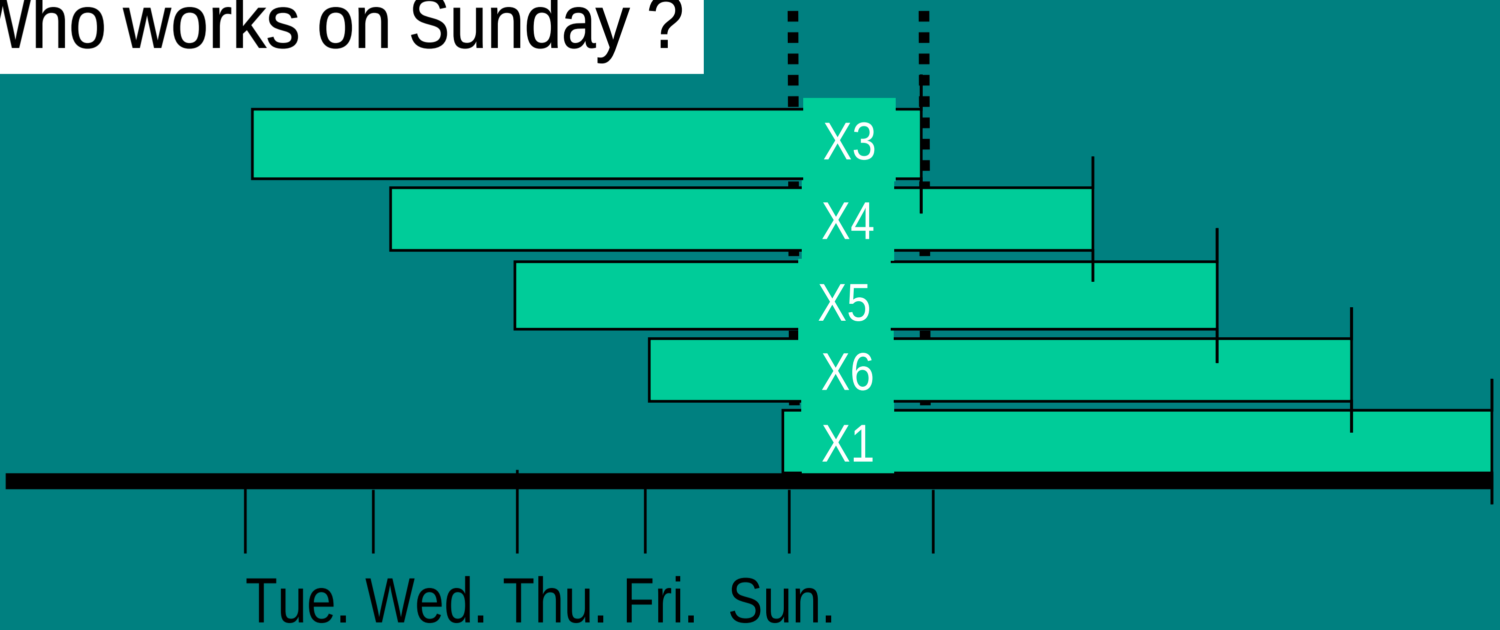
- **Decision Variables:**
  - X_i = the number of lifeguards scheduled to

    begin on day "I" for i=1, 2, ….,7 (i=1 is Sunday)

- **Objective Function:**
  - Minimize the total number of lifeguard scheduled

To ensure that enough lifeguards are scheduled for each day, ask which workers are on duty. For example:

Who works on Sunday ?



Repeat this procedure for each day of the week, and build the constraints accordingly.

- The Mathematical Model

Minimize $X1 + X2 + X3 + X4 + X5 + X6 + X7$

ST

$$X1 + X4 + X5 + X6 + X7 \geq 8 \quad \text{(Sunday)}$$

$$X1 + X2 + X5 + X6 + X7 \geq 6 \quad \text{(Monday)}$$

$$X1 + X2 + X3 + X6 + X7 \geq 5 \quad \text{(Tuesday)}$$

$$X1 + X2 + X3 + X4 + X7 \geq 4 \quad \text{(Wendnesday)}$$

$$X1 + X2 + X3 + X4 + X5 \geq 6 \quad \text{(Thursday)}$$

$$X2 + X3 + X4 + X5 + X6 \geq 7 \quad \text{(Friday)}$$

$$X3 + X4 + X5 + X6 + X7 \geq 9 \quad \text{(Saturday)}$$

All variables are non negative integers

POSSIBLE SUNSET BEACH LEFEGURAD ASSIGNMENTS

DAY	LIFEGUARDS PRESENT	REQUIRED	BEGIN SHIFT
SUNDAY	9	8	1
MONDAY	8	6	0
TUESDAY	6	5	1
WEDNESDAY	5	4	1
THURSDAY	6	6	3
FRIDAY	7	7	2
SATURDAY	9	9	2

TOTAL LIFEGUARDS 10

Note: An alternate optimal solution exists.

5.7 Binary Integer Linear Programming

- Binary variables take on only the values 0 or 1.
- Any situation that can be modeled by "yes"/"no", "good"/"bad" etc., falls into the binary category.
- Examples

$$X = \begin{cases} 1 & \text{If a new health care plan is adopted} \\ 0 & \text{If it is not} \end{cases}$$

$$X = \begin{cases} 1 & \text{If a new police station is built downtown} \\ 0 & \text{If it is not} \end{cases}$$

$$X = \begin{cases} 1 & \text{If a particular constraint must hold} \\ 0 & \text{If it is not} \end{cases}$$

SALEM CITY COUNCIL

- The Salem City Council must choose projects to fund, such that public support is maximized
- Relevant data covers constraints and concerns the City Council has, such as:
  - Estimated costs of each project.
  - Estimated number of permanent new jobs a project can create.
  - Questionnaire point tallies regarding the 9 project ranking.

Salem City Council - Funds Allocation

- The Salem City Council must choose projects to fund, such that public support is maximized while staying within a set of constraints and answering some concerns.

Survey results

- Data:

	Project		Cost (1000)	Jobs	Points
X1	Hire seven new police officers	$	400.00	7	4176
X2	Modernize police headquarters	$	350.00	0	1774
X3	Buy two new police cars	$	50.00	1	2513
X4	Give bonuses to foot patrol officers	$	100.00	0	1928
X5	Buy new fire truck/support equipment	$	500.00	2	3607
X6	Hire assistant fire chief	$	90.00	1	962
X7	Restore cuts to sport programs	$	220.00	8	2829
X8	Restore cuts to school music	$	150.00	3	1708
X9	Buy new computers for high school	$	140.00	2	3003

- Decision Variables:
  - X_j- a set of binary variables indicating if a project j is selected (X_j=1) or not (X_j=0) for j=1,2,..,9.
- Objective function:
  - Maximize the overall point score of the funded projects
- Constraints:
  - See the mathematical model.

- # The Mathematical Model

Max $4176X1 + 1774X2 + 2513X3 + 1928X4 + 3607X5 + 962X6 + 2829X7 + 1708X8 + 3003X9$

ST

The maximum amounts of funds to be allocated is $900,000

$$400X1 + 350X2 + 50X3 + 100X4 + 500X5 + 90X6 + 220X7 + 50X8 + 140X9 \leq 900$$

The number of new jobs created must be at least 10

$$7X1 + X3 + 2X5 + X6 + 8X7 + 3X8 + 2X9 \geq 10$$

The number of police-related activities selected is at most 3 (out of 4)

$$X1 + X2 + X3 + X4 \leq 3$$

Either police car or fire truck be purchased

$$X3 + X5 = 1$$

Sports funds and music funds must be restored / not restored together

$$X7 - X8 = 0$$

Sports funds and music funds must be restored before computer equipment is purchased

$$X7 - X9 \geq 0$$

$$X8 - X9 \geq 0$$

CONTINUE

Three of the following 5 constraints must be satisfied:

At least $250,000 must be reserved (do not use more than $650,000)

$$400X1 + 350X2 + 50X3 + 100X4 + 500X5 + 90X6 + 220X7 + 50X8 + 140X9 \leq 650$$

At least three police and fire stations should be funded

$$X1 + X2 + X3 + X4 + X5 + X6 \geq 3$$

Must hire seven new police officers

$$X1 = 1$$

At least fifteen new jobs should be created (not 10)

$$7X1 + X3 + 2X5 + X6 + 8X7 + 3X8 + 2X9 \geq 15$$

Three education projects should be funded

$$X7 + X8 + X9 = 3$$

The condition that at least three of these objectives are to be met can be expressed by the binary variable

$$Yi = \begin{cases} 1 & \text{if constraint i is ignored (the objective is not met)} \\ 0 & \text{If constraint i is not ignored (the objective is met)} \end{cases}$$

CONTINUE

This set of constraints is now added to the original model.

5.8 The Fixed Charge Location Problem

- Linear programming does not include fixed costs in its cost considerations. It assumes these costs it cannot be avoided. This may be incorrect.
- In the Fixed Charge Problem we have:

$$\text{Total Cost} = \begin{cases} CX + F & \text{If } X > 0 \\ 0 & \text{If } X = 0 \end{cases}$$

where:

C is a variable cost, and F is a fixed cost

GLOBE ELECTRONICS, INC.

- Globe Electronics, Inc. manufactures two styles of remote control cable boxes, G50 and G90.
- Globe runs four production facilities and three distribution centers.
- Each plant operates under unique conditions, thus has a different fixed operating cost, production costs, production rate, and production time available.

- Demand has decreased, therefore, management is contemplating closing one or more of its facilities.

- Management wishes to:
  - Develop an optimal distribution policy.
  - Determine which plant to close (if any).

- Data

Production costs, Times, Availability

Plant	Fixed Cost per Month	Production Cost per 100 G50	Production Cost per 100 G90	Production Time (hr/100) G50	Production Time (hr/100) G90	Available hr per Month
Philadelphia	40	1000	1400	6	6	640
St. Louis	35	1200	1200	7	8	960
New Orleans	20	800	1000	9	7	480
Denver	30	1300	1500	5	9	640

Monthly Demand Projection

	Demand Cincinnati	Demand Kansas City	Demand San Francisco
G50	2000	3000	5000
G90	5000	6000	7000

– Transportation Costs per 100 units

	Cincinnati	Kansas City	San Francisco
Philadelphia	$200	300	500
St.Louis	100	100	400
New Orleans	200	200	300
Denver	300	100	100

– At least 70% of the demand in each distribution center must be satisfied.

– Unit selling price

- G50 = $22;   G90 = $28.

- Decision Variables

  X_i = hundreds of G50s produced at plant i

  Z_i = hundreds of G90s produced at plant i

  X_{ij} = hundreds of G50s shipped from plant i to distribution center j

  Z_{ij} = hundreds of G90s shipped from plant i to distribution center j

Location Identification

Plant		Distribution Center	
Location	i	Location	j
Philadelphia	1	Cincinnati	1
St.Louis	2	Kansas City	2
New Orleans	3	San Francisco	3
Denver	4		

Globe Electronics
Model No. 1:
All The Plants Remain Operational

- Objective function
  - Management wants to maximize net profit.
  - Gross profit per 100 = 22(100) [minus] (production cost per 100)
  - Net profit per 100 units produced at plant i and shipped to center j = [Gross profit] -[Transportation cost from to j per 100]
  - Max 1200X1+1000X2+1400X3+ 900X4

+1400Z1+1600Z2+1800Z3+1300Z4

- 200X11 - 300X12 - 500X13

- 100X21 - 100X22 - 400X23

- 200X31 - 200X32 - 300X33

- 300X41 - 100X42 - 100X43

- 200Z11 - 300Z12 - 500Z13

- 100Z21 - 100Z22 - 400Z23

- 200Z31 - 200Z32 - 300Z33

- 300Z41 - 100Z42 - 100Z43

Gross profit

G50

Transportation cost

G90

- Constraints:

Production time used at each plant cannot exceed the time available:

$$6X1 + 6Z1 \leq 640$$

$$7X2 + 8Z2 \leq 960$$

$$9X3 + 7Z3 \leq 480$$

$$5X4 + 9Z4 \leq 640$$

All the variables are non negative

Linear and Integer Progra | Voice Menu | Sleeping | [Go to Sleep]

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Combined Report for GLOBE ELECTRONICS

	Decision Variable	Solution Value	Unit Cost or Profit c(j)	Total Contribution	Reduced Cost	Basis Status	Allowable Min. c(j)	Allowable Max. c(j)
1	X1	40.2041	1,200.0000	48,244.9000	0	basic	1,200.0000	1,200.0000
2	X2	9.7959	1,000.0000	9,795.9190	0	basic	1,000.0000	1,000.0000
3	X3	0	1,400.0000	0	0	basic	-M	1,485.7140
4	X4	50.0000	900.0000	45,000.0000	0	basic	814.2857	1,000.0000
5	Z1	0	1,400.0000	0	-300.0000	at bound	-M	1,700.0000
6	Z2	111.4286	1,600.0000	178,285.7000	0	basic	1,600.0000	1,666.6670
7	Z3	68.5714	1,800.0000	123,428.6000	0	basic	1,733.3330	M
8	Z4	0	1,300.0000	0	0	at bound	-M	1,300.0000
9	X11	20.0000	-200.0000	-4,000.0000	0	basic	-300.0000	M
10	X12	20.2041	-300.0000	-6,061.2250	0	basic	-300.0000	-300.0000
11	X13	0	-500.0000	0	-100.0000	at bound	-M	-400.0000
12	X	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	0
30	Z41	0	-300.0000	0	-500.0000	at bound	-M	200.0000
31	Z42	0	-100.0000	0	-300.0000	at bound	-M	200.0000
32	Z43	0	-100.0000	0	0	basic	-400.0000	-100.0000
	Objective	Function	(Max.) =	356,571.4000	(Note:	Alternate	Solution	Exists!!)

A portion of the WINQSB optimal solution

Results | Combined Report for GLOBE ELECTRONICS

Summary

- The optimal value of the objective function is $356,571.
- Note that the fixed cost of operating the plants was not included in the objective function because all the plants remain operational.
- Subtracting the fixed cost of $125,000 results in a net monthly profit of $231,571

Globe Electronics Model No. 2: The number of plants that remain operational in each city is a decision variable.

- Decision Variables

  X_i = hundreds of G50 s produced at plant i

  Z_i = hundreds of G90 s produced at plant i

  X_{ij} = hundreds of G50 s shipped from plant i to distribution center j

  Z_{ij} = hundreds of G90 s shipped from plant i to distribution center j

  Y_i = A 0-1 variable that describes the number of operational plants in city i.

- Objective function
  - Management wants to maximize net profit.
  - Gross profit per 100 = 22(100) - (production cost per 100)
  - Net profit per 100 produced at plant i and shipped to center j =

Gross profit - Costs of transportation from i to j - *Conditional fixed costs*

- Objective function

Max 1200X1+1000X2+1400X3+ 900X4

+1400Z1+1600Z2+1800Z3+1300Z4

- 200X11 - 300X12 - 500X13

- 100X21 - 100X22 - 400X23

- 200X31 - 200X32 - 300X33

- 300X41 - 100X42 - 100X43

- 200Z11 - 300Z12 - 500Z13

- 100Z21 - 100Z22 - 400Z23

- 200Z31 - 200Z32 - 300Z33

- 300Z41 - 100Z42 - 100Z43

- 40000Y1 - 35000Y2 - 20000Y3 - 30000Y4

- Constraints:

Ensure that the amount shipped from a plant equals the amount produced in a plant

Production time used at each plant cannot exceed the time available:

$$6X1 + 6Z1 - 640Y1 \leq 0$$

$$7X2 + 8Z2 - 960Y2 \leq 0$$

$$9X3 + 7Z3 - 480Y3 \leq 0$$

$$5X4 + 9Z4 - 640Y4 \leq 0$$

All X_{ij}, X_i, Z_{ij}, Z_i > 0, and Y_i are 0,1.

Linear and Integer Progra

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Combined Report for GLOBE ELECTRONICS

					30	1997
20	X43	50.0000	-100.0000	-5,000.0000	0	basic
21	Z11	0	-200.0000	0	0	basic
22	Z12	0	-300.0000	0	-100.0000	at bound
23	Z13	0	-500.0000	0	-107.1429	at bound
24	Z21	50.0000	-100.0000	-5,000.0000	0	basic
25	Z22	60.0000	-100.0000	-6,000.0000	0	basic
26	Z23	0	-400.0000	0	-107.1429	at bound
27	Z31	0	-200.0000	0	-92.8571	at bound
28	Z32	0	-200.0000	0	-92.8571	at bound
29	Z33	32.5155	-300.0000	-9,754.6590	0	basic
32	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
33						
34						
35	Y3	1.0000	-20,000.0000	-20,000.0000	0	basic
36	Y4	1.0000	-30,000.0000	-30,000.0000	-30,000.0000	at bound
	Objective	Function	(Max.) =	266,115.0000		
	Constraint	Left Hand Side	Direction	Right Hand Side	Slack or Surplus	Shadow Price

A portion of the WINQSB optimal solution

Results | Combined Report for GLOBE ELECTRONICS

Summary

- The Philadelphia plant should be closed.
- Schedule monthly production according to the quantities shown in the output.
- The net monthly profit will be $266,115, which is $34,544 per month greater than the optimal monthly profit obtained when all four plants are operational.