Chapter 6

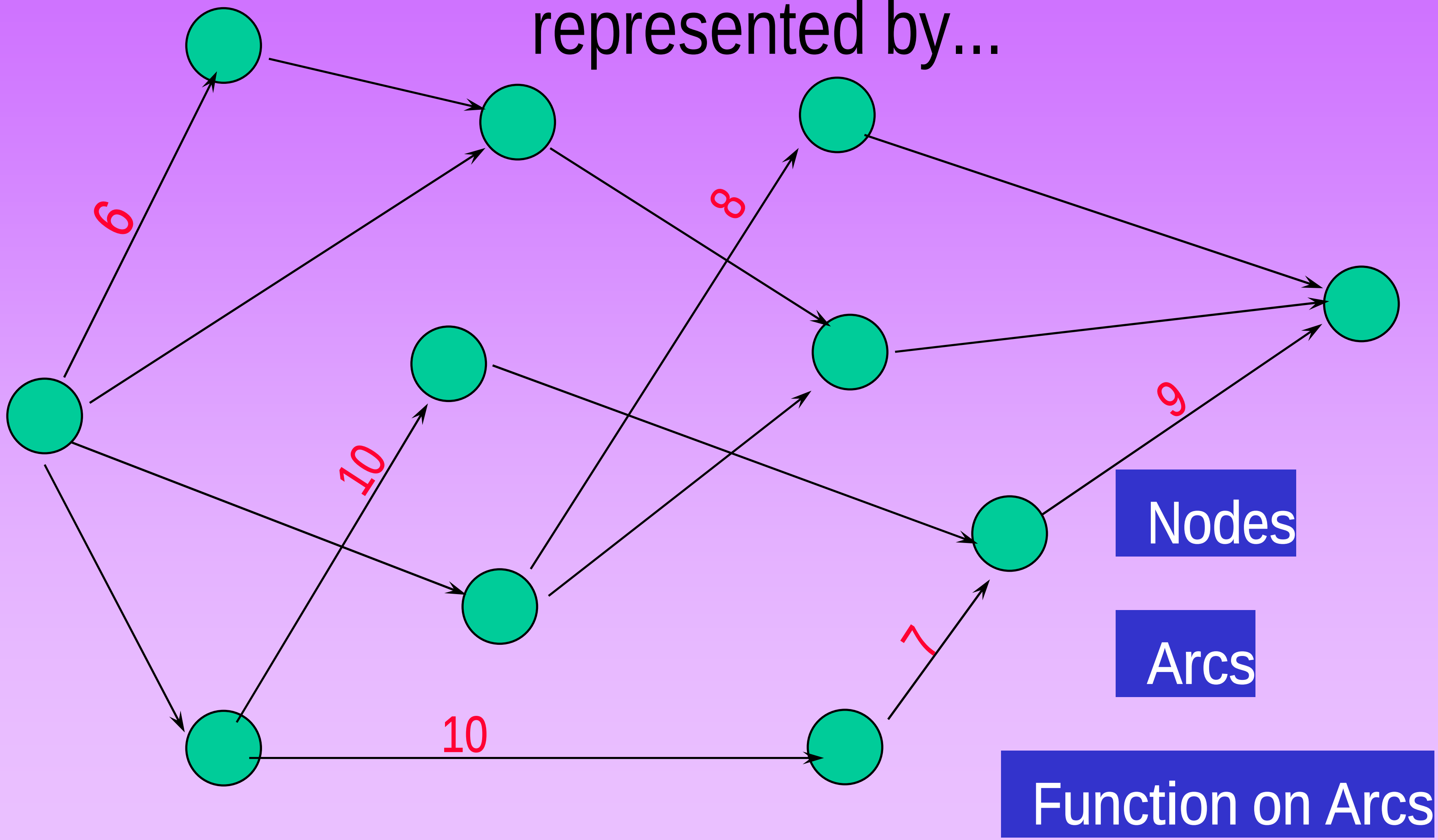
Network

Models

Chapter Objectives

- Network concepts and definitions.
- Importance of network models.
- Linear programming models, network representations, and computer solutions for
  - Transportation models.
  - Capacitated transshipment models.
  - Assignment models.
  - Travelling salesman models.
  - Shortest path models.
  - Minimal spanning tree models.
  - Maximum flow models.

A network problem is one that can be represented by...



6.1 Introduction

- The importance of network models
  - Many business problems lend themselves to a network formulation.
  - Optimal solutions of network problems are guaranteed integer solutions, because of special mathematical structures. No special restrictions are needed to ensure integrality
  - Network problems can be efficiently solved by compact algorithms due to there special mathematical structure, even for large scale models.

- Network Terminology

– *Flow* : the amount sent from node i to node j, over an arc that connects them. The following notation is used:

X_{ij} = amount of flow

U_{ij} = upper bound of the flow

L_{ij} = lower bound of the flow

– *Directed/undirected arcs* : when flow is allowed in one direction the arc is directed (marked by an arrow). When flow is allowed in two directions, the arc is undirected (no arrows).

– *Adjacent nodes* : a node (j) is adjacent to another node (i) if an arc joins node i to node j.

- Path / Connected nodes
  - *Path* :a collection of arcs formed by a series of adjacent nodes.
  - The nodes are said to be connected if there is a path between them.

- Cycles / Trees / Spanning Trees
  - *Cycle* : a path starting at a certain node and returning to the same node without using any arc twice.
  - *Tree* : a series of nodes that contain no cycles.
  - *Spanning tree* : a tree that connects all the nodes in a network (it consists of n -1 arcs).

6.2 The Transportation Problem

Transportation problems arise when a cost-effective pattern is needed to ship items from origins that have limited supply to destinations that have demand for the goods.

- **Problem definition**

  - There are m sources. Source i has a supply capacity of S_i.

  - There are n destinations. The demand at destination j is D_j.

  - Objective:

    Minimize the total shipping cost of supplying the destinations with the required demand from the available supplies at the sources.

CARLTON PHARMACEUTICALS

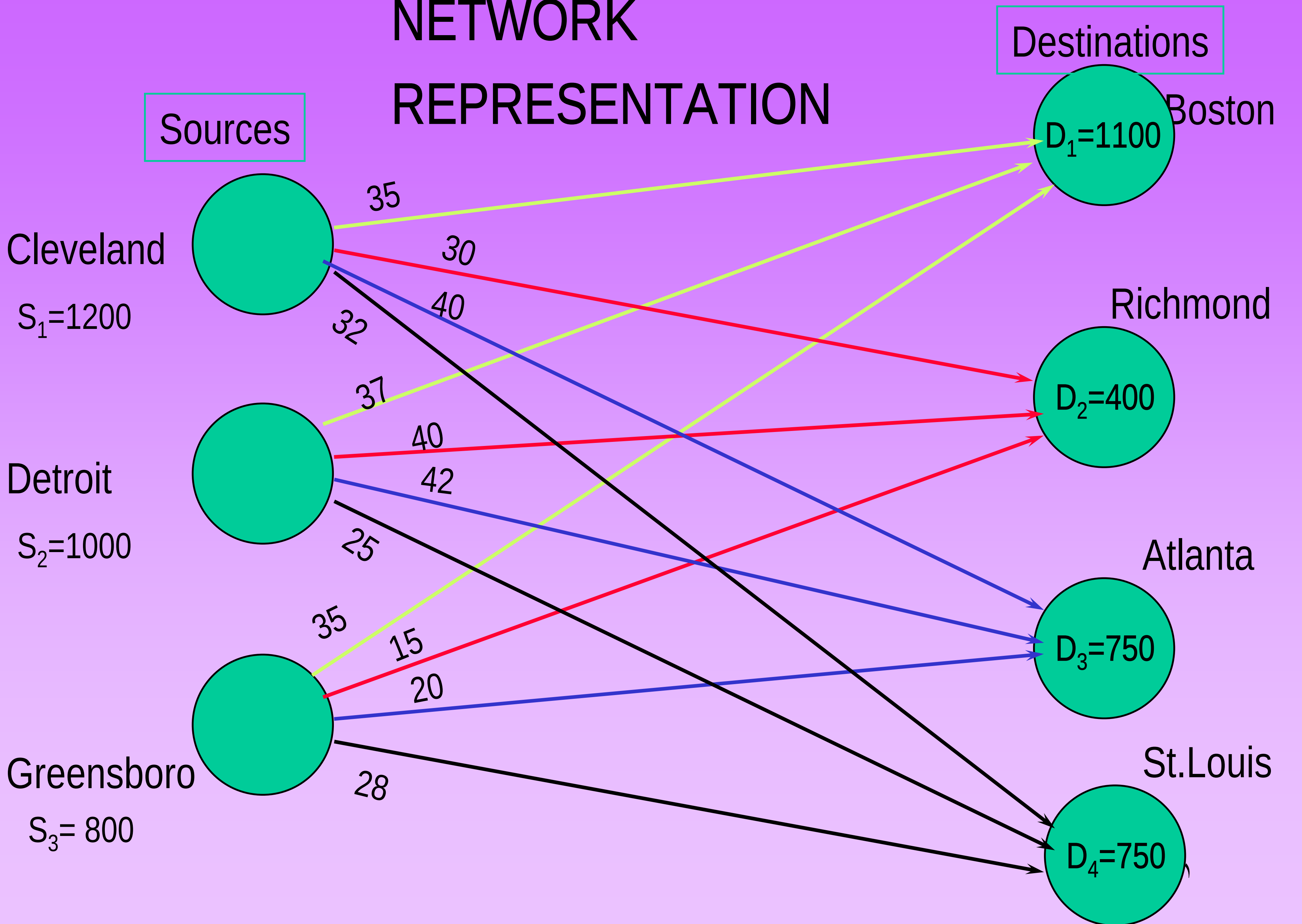
- Carlton Pharmaceuticals supplies drugs and other medical supplies.
- It has three plants in: Cleveland, Detroit, Greensboro.
- It has four distribution centers in: Boston, Richmond, Atlanta, St. Louis.
- Management at Carlton would like to ship cases

- Data
  - Unit shipping cost, supply, and demand

From	To				Supply
	Boston	Richmond	Atlanta	St. Louis	
Cleveland	$35	30	40	32	1200
Detroit	37	40	42	25	1000
Greensboro	40	15	20	28	800
Demand	1100	400	750	750	

- Assumptions
  - Unit shipping cost is constant.
  - All the shipping occurs simultaneously.
  - The only transportation considered is between sources and destinations.
  - Total supply equals total demand.

NETWORK REPRESENTATION



- # The Mathematical Model

– The structure of the model is:

Minimize <Total Shipping Cost>

ST

[Amount shipped from a source] = [Supply at that source]

[Amount received at a destination] = [Demand at that destination]

– Decision variables

X_{ij} = amount shipped from source i to destination j.

where: i=1 (Cleveland), 2 (Detroit), 3 (Greensboro)

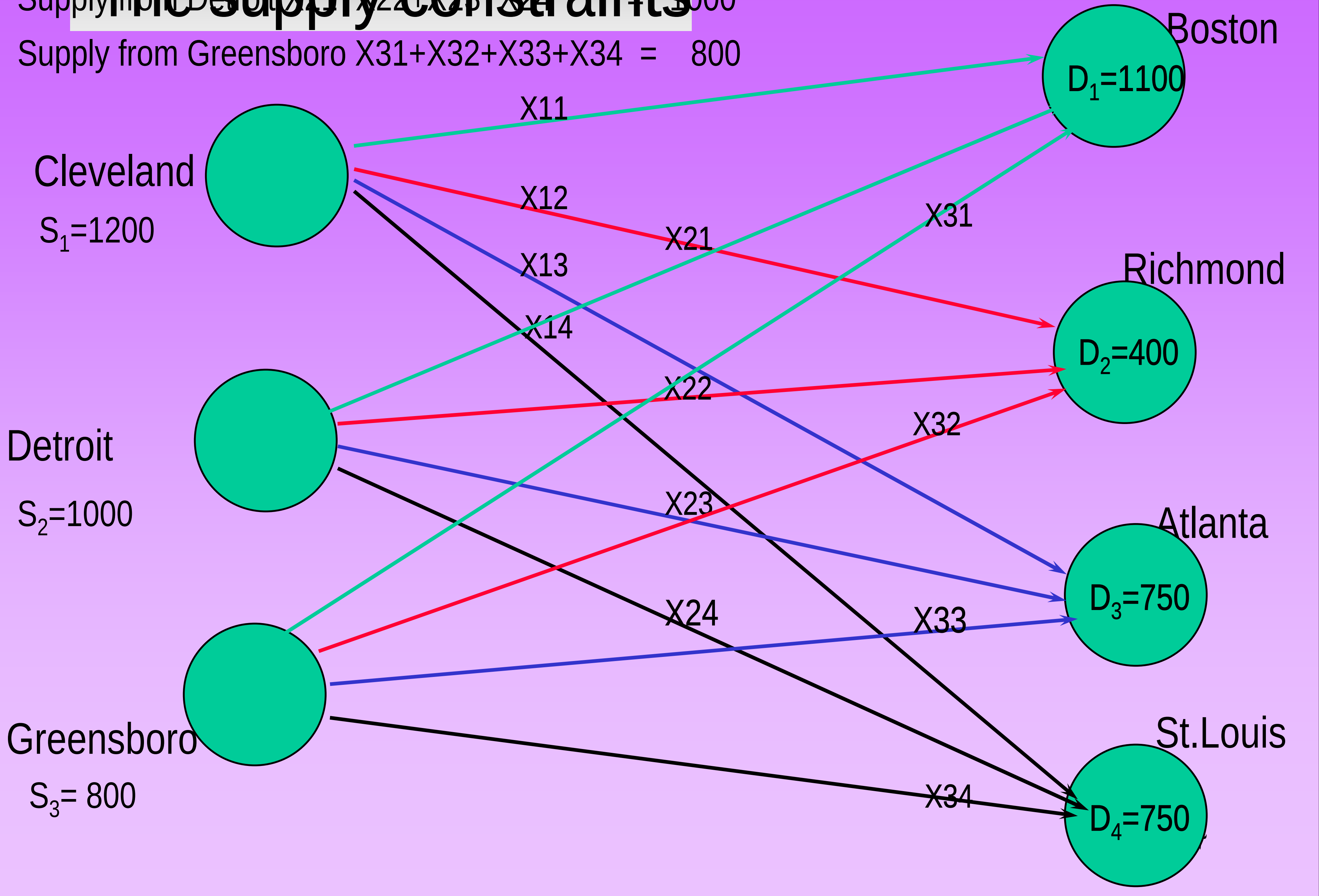
j=1 (Boston), 2 (Richmond), 3 (Atlanta), 4(St.Louis)

The supply constraints

Supply from Cleveland X11+X12+X13+X14 = 1200

Supply from Detroit X21+X22+X23+X24 = 1000

Supply from Greensboro X31+X32+X33+X34 = 800



- The complete mathematical model

Minimize 35X11+30X12+40X13+ 32X14 +37X21+40X22+42X23+25X24+
40X31+15X32+20X33+38X34

ST

Supply constrraints:

$$X_{11}+X_{12}+X_{13}+X_{14} = 1200$$
$$X_{21}+X_{22}+X_{23}+X_{24} = 1000$$
$$X_{31}+X_{32}+X_{33}+X_{34} = 800$$

Demand constraints:

$$X_{11}+X_{21}+X_{31} = 1000$$
$$X_{12}+X_{22}+X_{32} = 400$$
$$X_{13}+X_{23}+X_{33} = 750$$
$$X_{14}+X_{24}+X_{34} = 750$$

All Xij are nonnegative

Excel Optimal Solution

		CARLTON PHARMACEUTICALS					
		UNIT COSTS					
	BOSTON	RICHMOND	ATLANTA	ST.LOUIS		SUPPLIES	
CLEVELAND	$ 35.00	$ 30.00	$ 40.00	$ 32.00		1200	
DETROIT	$ 37.00	$ 40.00	$ 42.00	$ 25.00		1000	
GREENSBORO	$ 40.00	$ 15.00	$ 20.00	$ 28.00		800	
DEMANDS	1100	400	750	750			
		SHIPMENTS (CASES)					
	BOSTON	RICHMOND	ATLANTA	ST.LOUIS		TOTAL	
CLEVELAND	850	350	0	0		1200	
DETROIT	250	0	0	750		1000	
GREENSBORO	0	50	750	0		800	
TOTAL	1100	400	750	750			
					TOTAL	COST =	84000

WINQSB Sensitivity Analysis

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Range of Optimality for CARLTON PHARMACEUTICALS: Minimization (Transportation

07-05-1997 19:28:03	From	To	Unit Cost	Reduced Cost	Basis Status	Allowable Min. Cost	Allowable Max. Cost
1	CLEVELAND	BOSTON	35	0	basic	30	37
2	CLEVELAND	RICHMOND	30	0	basic	13	35
3	CLEVELAND	ATLANTA	40	5	at bound	35	M
4	CLEVELAND	ST. LOUIS	32	9	at bound	23	M
5	DETROIT	BOSTON	37	0	basic	35	42
6	DETROIT	RICHMOND	40	8	at bound	32	M
7	DETROIT	ATLANTA	42	5	at bound	37	M
8	DETROIT	ST. LOUIS	25	0	basic	0	34
9	GREENSBORO	BOSTON	40	20	at bound	20	M
10	GREENSBORO	RICHMOND	15	0	basic	10	32
11	GREENSBORO	ATLANTA	20	0	basic	-17	25
12	GREENSBORO	ST. LOUIS	28	20	at bound	8	M

If this path is used, the total cost will increase by $5 per unit shipped along it

Range of optimality

Results Print

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Range of feasibility

Range of Feasibility for CARLTON PHARMACEUTICA[illegible]ation (Transportation

	Node	Supply	Demand	Shadow Price	Mi[illegible]	Allowable Max. Value
1	CLEVELAND	1200	0	-2	1200	1450
2	DETROIT	1000	0	0	1000	M
3	GREENSBORO	800	0	-17	800	1050
4	BOSTON	0	1100	37	850	1100
5	RICHMOND	0	400	32	150	400
6	ATLANTA	0	750	37	500	750
7	ST. LOUIS	0	750	25	0	750

Shadow prices for warehouses - the cost saving resulting from 1 extra case of vaccine demanded at the warehouse

Shadow prices for plants - the cost incurred for each extra case of vaccine available at the plant

Results Paste

- Interpreting sensitivity analysis results
  - Reduced costs
    - The amount of transportation cost reduction per unit that makes a given route economically attractive.
    - If the route is forced to be used under the current cost structure, for each item shipped along it, the total cost increases by an amount equal to the reduced cost.
  - Shadow prices
    - For the plants, shadow prices convey the cost savings realized for each extra case of vaccine available at plant.
    - For the warehouses, shadow prices convey the cost incurred from having an extra case demanded at the warehouse.

- **Special cases of the transportation problem**
  - Cases may arise that appear to violate the assumptions necessary to solve the transportation problem using standard methods.
  - Modifying the resulting models make it possible to use standard solution methods.
  - Examples:
    - Blocked routes - shipments along certain routes are prohibited.
    - Minimum shipment - the amount shipped along a certain route must not fall below a prespecified level.
    - Maximum shipment - an upper limit is placed on the amount shipped along a certain route.
    - Transshipment nodes - intermediate nodes that may have demand , supply, or no demand and no supply of their own.

  - General network problems are solved by the *"Out-of-Kilter"* algorithm.

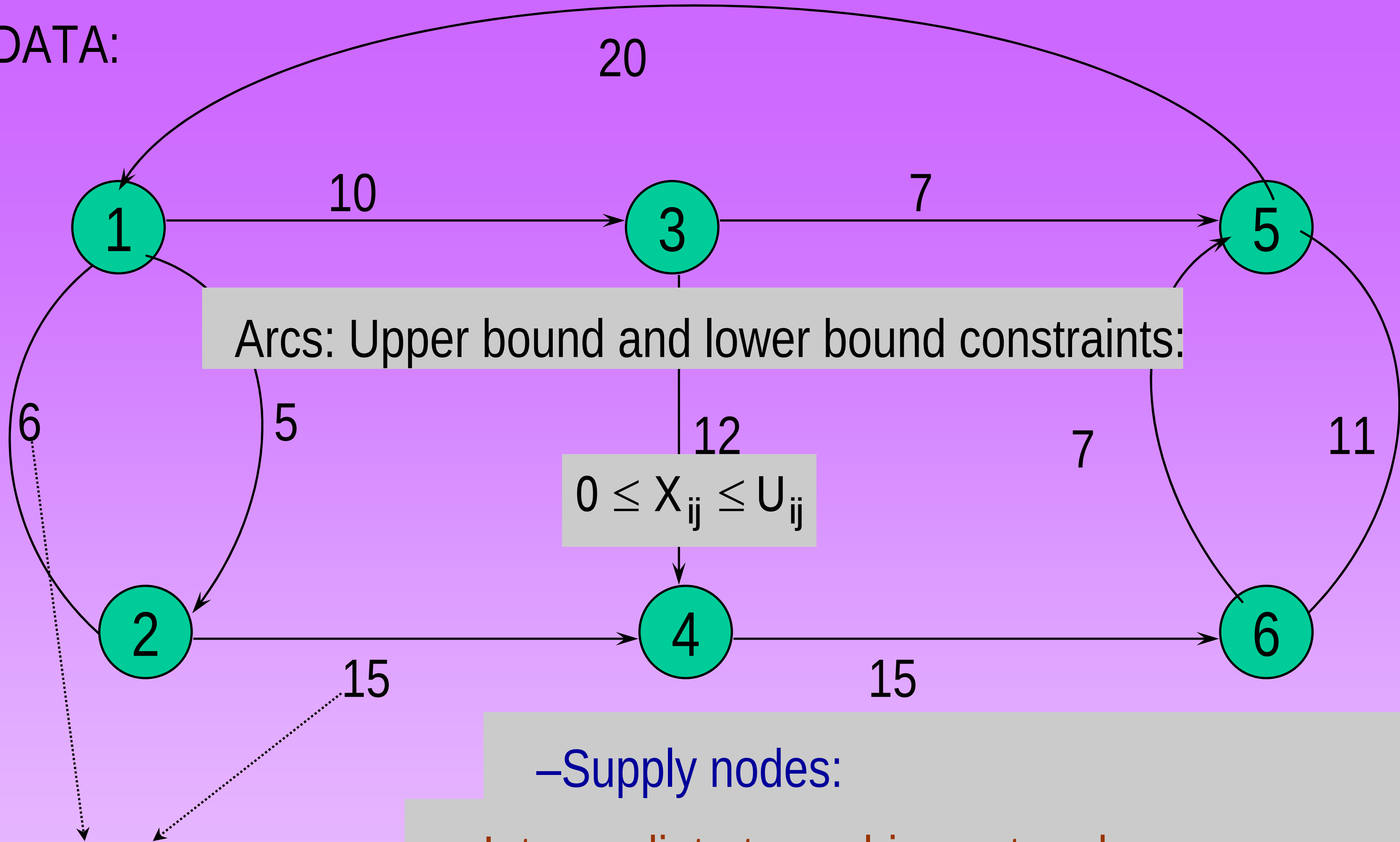
DEPOT MAX
A General Network Problem

- Depot Max has six stores.
  - Stores 5 and 6 are running low on the model 65A Arcadia workstation, and need a total of 25 additional units.
  - Stores 1 and 2 are ordered to ship a total of 25 units to stores 5 and 6.
  - Stores 3 and 4 are transshipment nodes with no demand or supply of their own.

- Other restrictions
  - There is a maximum limit for quantities shipped on various routes.
  - There are different unit transportation costs for different routes.

- Depot Max wishes to transport the available workstations at minimum total cost.

- DATA:



Arcs: Upper bound and lower bound constraints:

$0 \leq X_{ij} \leq U_{ij}$

Transportation unit cost

– Supply nodes:

– Intermediate transshipment nodes:

– Demand nodes:

[Net flow into the node] = [Demand for the node]

X15 + X35 +X65 - X56 = 12 (Node 5)

X46 +X56 - X65 = 13 (Node 6)

- The Complete mathematical model

$$\text{Minimize } 5X12 + 10X13 + 20X15 + 6X21 + 15X24 + 12X34 + 7X35 + 15X46 + 11X56 + 7X65$$

ST

$$X12 + X13 + X15 - X21 = 10$$

$$-X12 + X21 + X24 = 15$$

$$-X13 + X34 + X35 = 0$$

$$-X24 - X34 + X46 = 0$$

$$-X15 - X35 + X56 - X65 = -12$$

$$-X46 - X56 + X65 = -13$$

$$0 \le X12 \le 3;\ 0 \le X13 \le 12;\ 0 \le X15 \le 6;\ 0 \le X21 \le 7;\ 0 \le X24 \le 10;\ 0 \le X34 \le 8;\ 0 \le X35 \le 8;$$

$$0 \le X46 \le 17;\ 0 \le X56 \le 7;\ 0 \le X65 \le 5$$

Network Modeling

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CAPACITATED TRANSSHIPMENT -- OFFICE DEPOT: Minimization (Network Flow Problem)

Node1 : Node1

From \ To	Node1	Node2	Node3	Node4	Node5	Node6	Supply
Node1		5	10		20		10
Node2	6			15			15
Node3				12	7		0
Node4						15	0
Node5						11	0
Node6					7		0
Demand	0	0	0	0	12	13	

WINQSB Input Data

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Network Modeling

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Solution for CAPACITATED TRANSSHIPMENT -- OFFICE DEPOT: Minimization (Network Flo...

	From	To	Flow	Unit Cost	Total Cost	Reduced Cost
1	Node1	Node3	9	10	90	0
2	Node1	Node5	6	20	120	-6
3	Node2	Node1	5	6	30	0
4	Node2	Node4	10	15	150	-13
5	Node3	Node4	1	12	12	0
6	Node3	Node5	8	7	56	-9
7	Node4	Node6	11	15	165	0
8	Node5	Node6	2	11	22	0
	Total	Objective	Function	Value =	645	

WINQSB Optimal Solution

Results Results Menu

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MONTPELIER SKI COMPANY

Using a Transportation model for production scheduling

- Montpelier is planning its production of skis for the months of July, August, and September.
- Production capacity and unit production cost will change from month to month.
- The company can use both regular time and overtime to produce skis.
- Production levels should meet both demand forecasts and end-of-quarter inventory requirement.
- Management would like to schedule production to minimize its costs for the quarter.

- Data:
  - Initial inventory = 200 pairs
  - Ending inventory required =1200 pairs
  - Production capacity for the next quarter = 400 pairs in regular time.
    = 200 pairs in overtime.
  - Holding cost rate is 3% per month per ski.
  - Production capacity, and forecasted demand for this quarter (in pairs of skis), and production cost per unit (by months)

Month	Forecasted Demand	Production Capacity	Production Costs Regular Time	Production Costs Overtime
July	400	1000	25	30
August	600	800	26	32
September	1000	400	29	37

- Analysis of demand:
  - Net demand to satisfy in July = 400 - 200 = 200 pairs

- Analysis of Unit costs

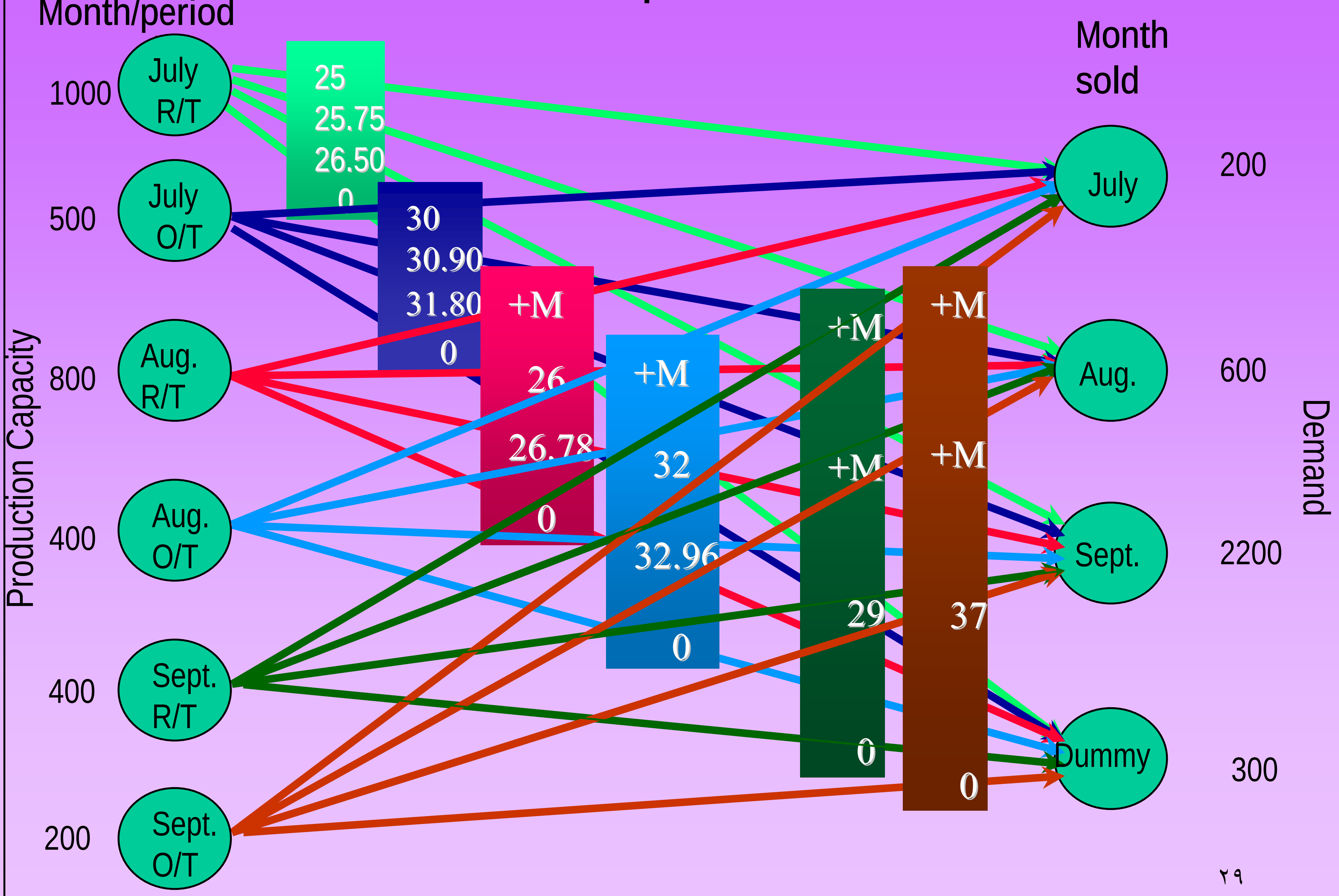
Unit cost = [Unit production cost] +

[Unit holding cost per month][the number of months stays in inventory]

Example: A unit produced in July in Regular time and sold in September costs 25+ (3%)(25)(2 months) = $26.50

- Set 1- Regular time supply (production capacity)
- Set 2 - Overtime supply

Network representation



Network Modeling

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MONTPELIER SKI COMPANY: Minimization (Transportation Problem)

JULY-REG : SOLD-JUL 25

From \ To	SOLD-JUL	SOLD-AUG	SOLD-SEP	Supply
JULY-REG	25	25.75	26.50	1000
JULY-O/T	30	30.90	31.80	500
AUG-REG		26	26.78	800
AUG-O/T		32	32.96	400
SEP-REG			29	400
SEP-O/T			37	200
Demand	200	600	2200	

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- **Summary of the optimal solution**
  - In July produce at capacity (1000 pairs in R/T, and 500 pairs in O/T). Store 1500-200 = 1300 at the end of July.
  - In August, produce 800 pairs in R/T, and 300 in O/T. Store additional 800 + 300 - 600 = 500 pairs.
  - In September, produce 400 pairs (clearly in R/T). With 1000 pairs retail demand, there will be

(1300 + 500) + 400 - 1000 = 1200 pairs available for shipment to Ski Chalet.

Inventory + Production - Demand

6.3 The Assignment Problem

- Problem definition
  - m workers are to be assigned to m jobs
  - A unit cost (or profit) C_{ij} is associated with worker i performing job j.
  - Minimize the total cost (or maximize the total profit) of assigning workers to job so that each worker is assigned a job, and each job is performed.

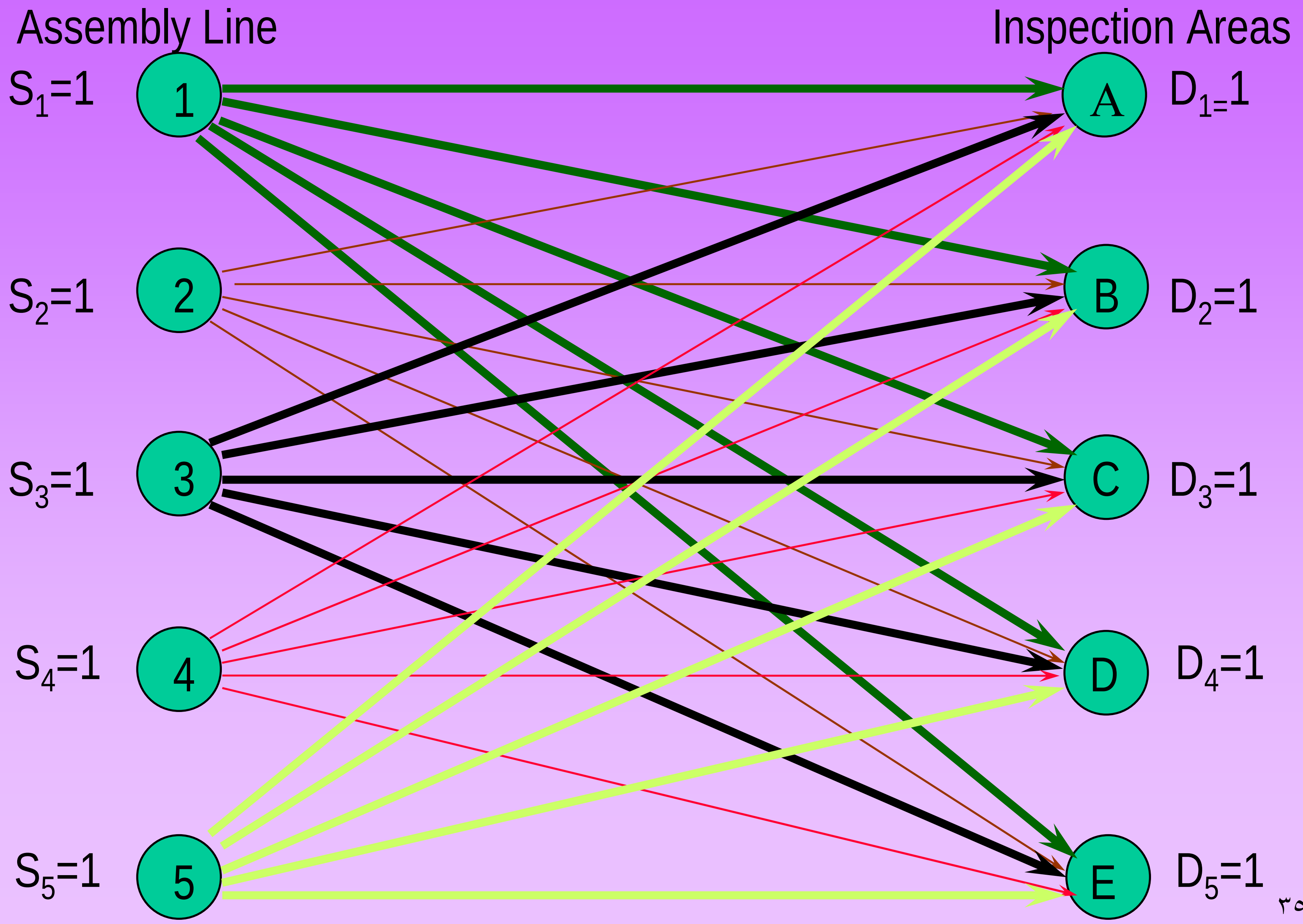
BALLSTON ELECTRONICS

- Five different electrical devices produced on five production lines, are needed to be inspected.
- The travel time of finished goods to inspection areas depends on both the production line and the inspection area.
- Management wishes to designate a separate inspection area to inspect the products such that the total travel time is minimized.

- Data: Travel time in minutes from assembly lines to inspection areas.

		Inspection Area				
		A	B	C	D	E
Assembly Lines	1	10	4	6	10	12
	2	11	7	7	9	14
	3	13	8	12	14	15
	4	14	16	13	17	17
	5	19	17	11	20	19

NETWORK REPRESENTATION



- Assumptions and restrictions
  - The number of workers equals the number of jobs.
  - Given a balanced problem, each worker is assigned exactly once, and each job is performed by exactly one worker.
  - For an unbalanced problem “dummy” workers (in case there are more jobs than workers), or “dummy” jobs (in case there are more workers than jobs) are added to balance the problem.

- Computer solutions
  - A complete enumeration is not efficient even for moderately large problems (with m=8, m! > 40,000 is the number of assignments to enumerate).
  - The *Hungarian method* provides an efficient solution procedure.

- Special cases
  - A worker is unable to perform a particular job.
  - A worker can be assigned to more than one job.
  - A maximization assignment problem.

6.4 The Traveling Salesman Problem

- Problem definition
  - There are m nodes.
  - Unit cost C_{ij} is associated with utilizing arc (i,j)
  - Find the cycle that minimizes the total cost required to visit all the nodes exactly once.

- Complexity

Writing the mathematical model and solving this problem are both cumbersome (a problem with 20 cities requires over 500,000 linear constraints.)

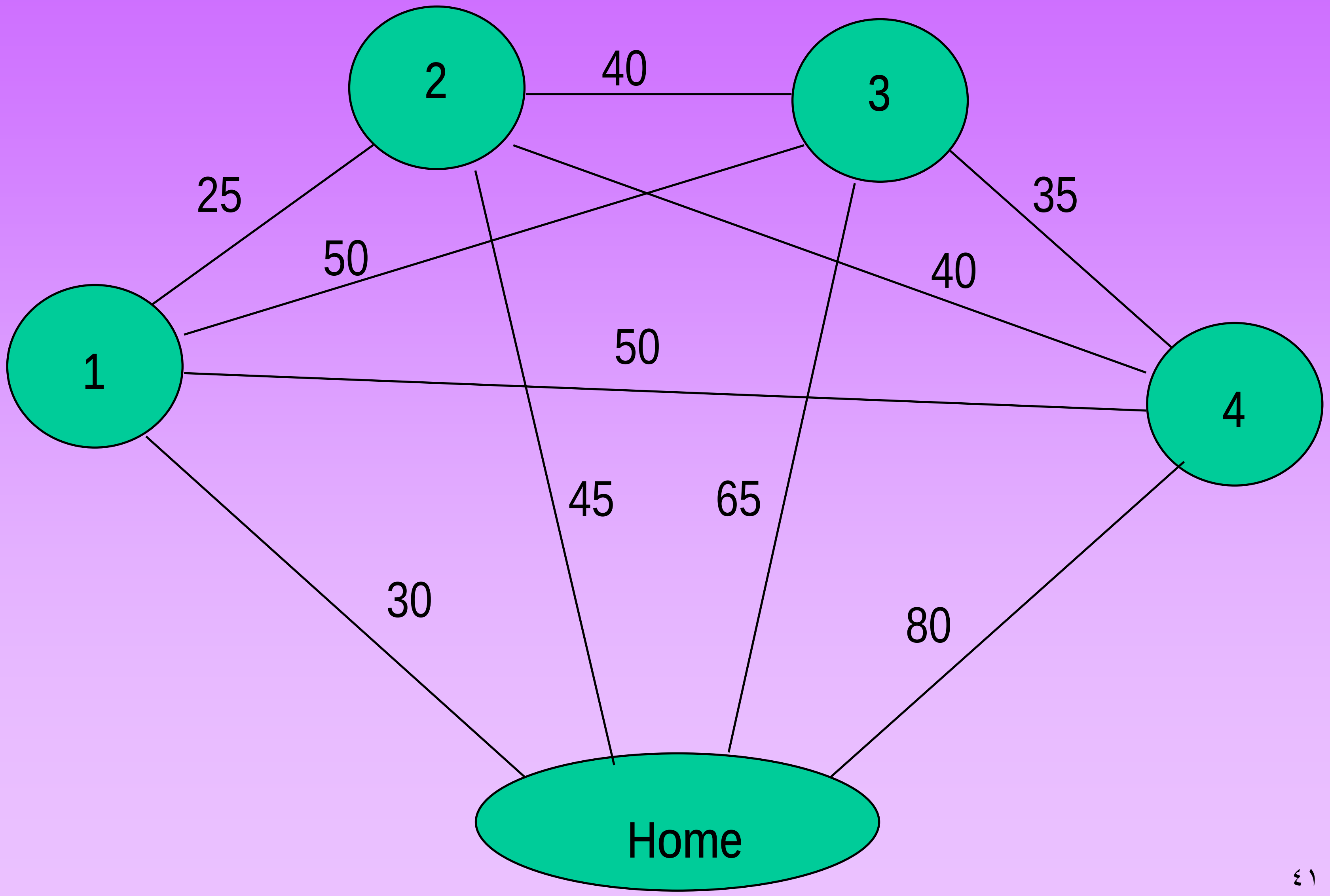
THE FEDERAL EMERGENCY MANAGEMENT AGENCY

- A visit must be made to four local offices of FEMA, going out from and returning to the same main office in Northridge, Southern California.
- Data

Travel time between offices (minutes)

		To office				
		H	1	2	3	4
F	Home office		30	45	65	80
r	Office 1	30		25	50	50
o	Office 2	45	25		40	40
m	Office 3	65	50	40		35
	Office 4	80	50	40	35	

FEMA traveling salesman network representation



- Solution approaches
  - Enumeration of all possible cycles.
    - This results in (m-1)! cycles to enumerate.
    - Only small problems can be solved with this approach.
  - A combination of the Assignment problem and the Branch and Bound technique.
    - Problem with up to m=20 nodes can be efficiently solved with this approach.

The FEMA problem - A full enumeration

For this problem we have (5-1)! / 2 = 12 cycles. Symmetrical problems have (m-1)! / 2 cycles to enumerate

Possible cycles

Cycle	Total Cost
1. H-O1-O2-O3-O4-H	210
2. H-O1-O2-O4-O3-H	195
3. H-O1-O3-O2-O3-H	240
4. H-O1-O3-O4-O2-H	200
5. H-O1-O4-O2-O3-H	225
6. H-O1-O4-O3-O2-H	200
7. H-O2-O3-O1-O4-H	265
8. H-O2-O1-O3-O4-H	235
9. H-O2-O4-O1-O3-H	250
10. H-O2-O1-O4-O3-H	220
11. H-O3-O1-O2-O4-H	260
12. H-O3-O1-O2-O4-H	260

Minimum

Network Modeling

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FEMA: Minimization (Traveling Salesman Problem)

HOME : HOME

From \ To	HOME	OFFICE 1	OFFICE 2	OFFICE 3	OFFICE 4
HOME		30	45	65	80
OFFICE 1	30		25	50	50
OFFICE 2	45	25		40	40
OFFICE 3	65	50	40		35
OFFICE 4	80	50	40	35	

WINQSB input data for the Traveling Salesman problem

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Solution for FEMA: Minimization (Traveling Salesman Problem)

	From Node	Connect To	Distance/Cost		From Node	Connect To	Distance/Cost
1	HOME	OFFICE 3	65	4	OFFICE 2	OFFICE 1	25
2	OFFICE 3	OFFICE 4	35	5	OFFICE 1	HOME	30
3	OFFICE 4	OFFICE 2	40				
	Total	Minimal	Traveling	Distance	or Cost	=	195
	(Result	from	Branch	and	Bound	Method)	

WINQSB Solution - by the combination of the Assignment problem and the Branch and Bound technique

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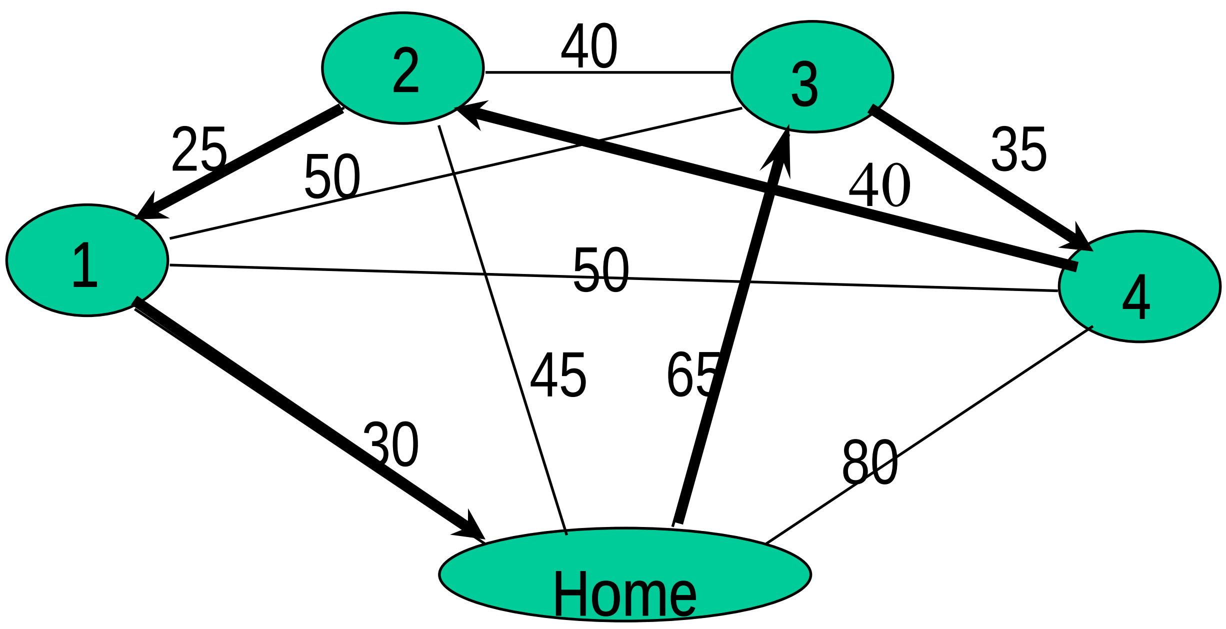
Solution for FEMA: Minimization (Traveling Salesman Problem)

	From Node	Connect To	Distance/Cost		From Node	Connect To	Distance/Cost
1	HOME	OFFICE 3	65	4	OFFICE 2	OFFICE 1	25
2	OFFICE 3	OFFICE 4	35	5	OFFICE 1	HOME	30
3	OFFICE 4	OFFICE 2	40				
	Total	Minimal	Traveling	Distance	or Cost	=	195
	(Result	from	Branch	and	Bound	Method)	



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- Special Cases

  – Revisiting nodes : a node can be revisited before the end of the cycle. To handle this situation:
    - find the shortest path from each city to any other city,
    - substitute the shortest path for the ”direct distance” value.
    - solve the traveling salesman problem with the new distances.

  – n-person traveling salesman problem
    - n objects must visit m nodes, but no two objects visit the same node. The objective is to minimize

      (1) the overall miles traveled, or

      (2) the maximum distance traveled, or

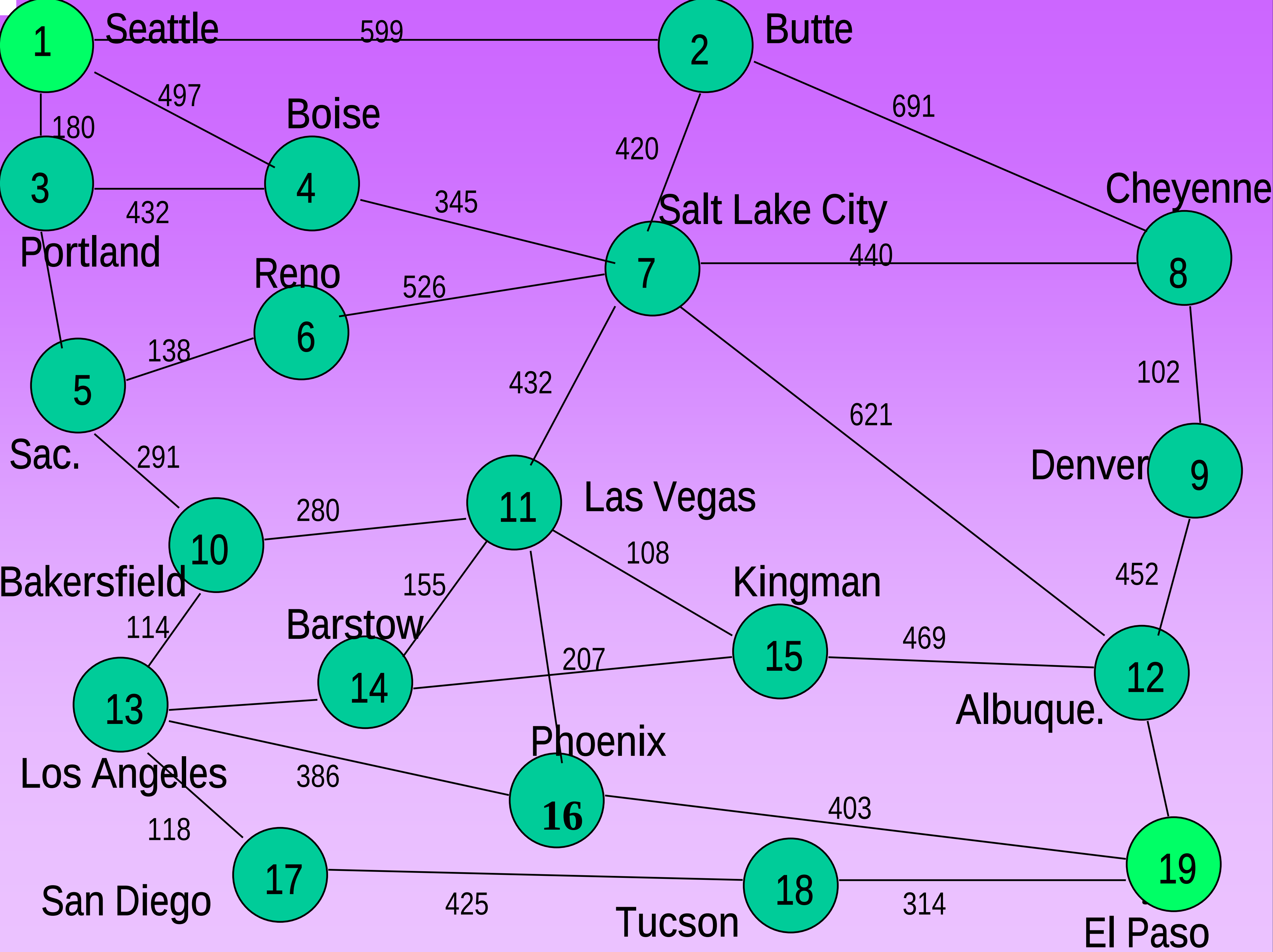
      (3) the total costs incurred.

6.5 The Shortest Path Problem

- For a given network find the path of minimum distance, time, or cost from a starting point, the *start node*, to a destination, the *terminal node*.
- Problem definition
  - There are n nodes, beginning with start node 1 and ending with terminal node n.
  - Bi-directional arcs connect connected nodes i and j with nonnegative distances, $d_{i\,j}$.
  - Find the path of minimum total distance that connects node 1 to node n.

Fairway Van Lines

Determine the shortest route from Seattle to El Paso over the following network highways.



1
Seattle
599
2
Butte
497
180
Boise
691
420
3
4
345
Cheyenne
Salt Lake City
432
Portland
440
7
8
Reno
526
6
138
102
5
432
621
Sac.
291
Denver
9
11
Las Vegas
280
10
108
Bakersfield
155
Kingman
452
114
Barstow
469
207
15
14
12
13
Albuque.
Phoenix
Los Angeles
386
16
403
118
17
18
19
San Diego
425
314
Tucson
El Paso

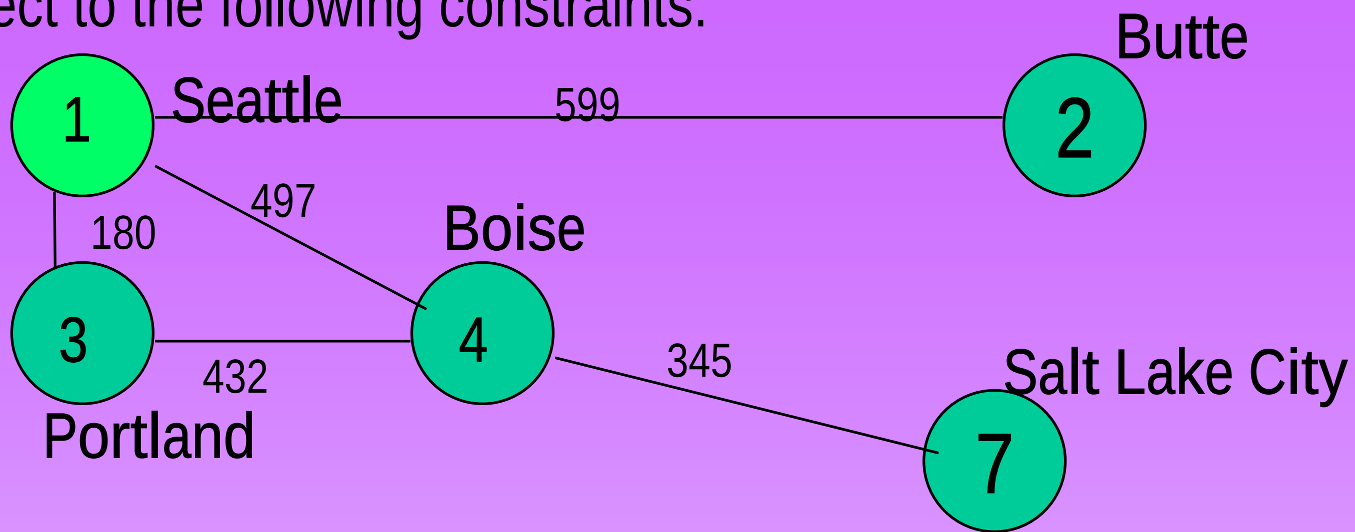
- **Solution - a linear programming approach**

Decision variables

$$X_{ij} = \begin{cases} 1 & \text{if a truck travels on the highway from city } i \text{ to city } j \\ 0 & \text{otherwise} \end{cases}$$

Objective = Minimize $\Sigma\, d_{ij}X_{ij}$

Subject to the following constraints:



[The number of highways traveled out of Seattle (the start node)] = 1

$X12 + X13 + X14 = 1$

In a similar manner:

[The number of highways traveled into El Paso (terminal node)] = 1

$X12,19 + X16,19 + X18,19 = 1$

[The number of highways used to travel into a city] = [The number of highways traveled leaving the city].

For example, in Boise (City 4):

$X14 + X34 + X74 = X41 + X43 + X47.$

Nonnegativity constraints

Network Modeling

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Solution for Shortest Path Problem FAIRWAY VAN LINES

	From	To	Distance/Cost	Cumulative Distance/Cost
1	SEA	BOI	497	497
2	BOI	SLC	345	842
3	SLC	ALB	621	1463
4	ALB	ELP	268	1731
	From SEA	To ELP	Distance/Cost	= 1731

WINQSB Optimal Solution

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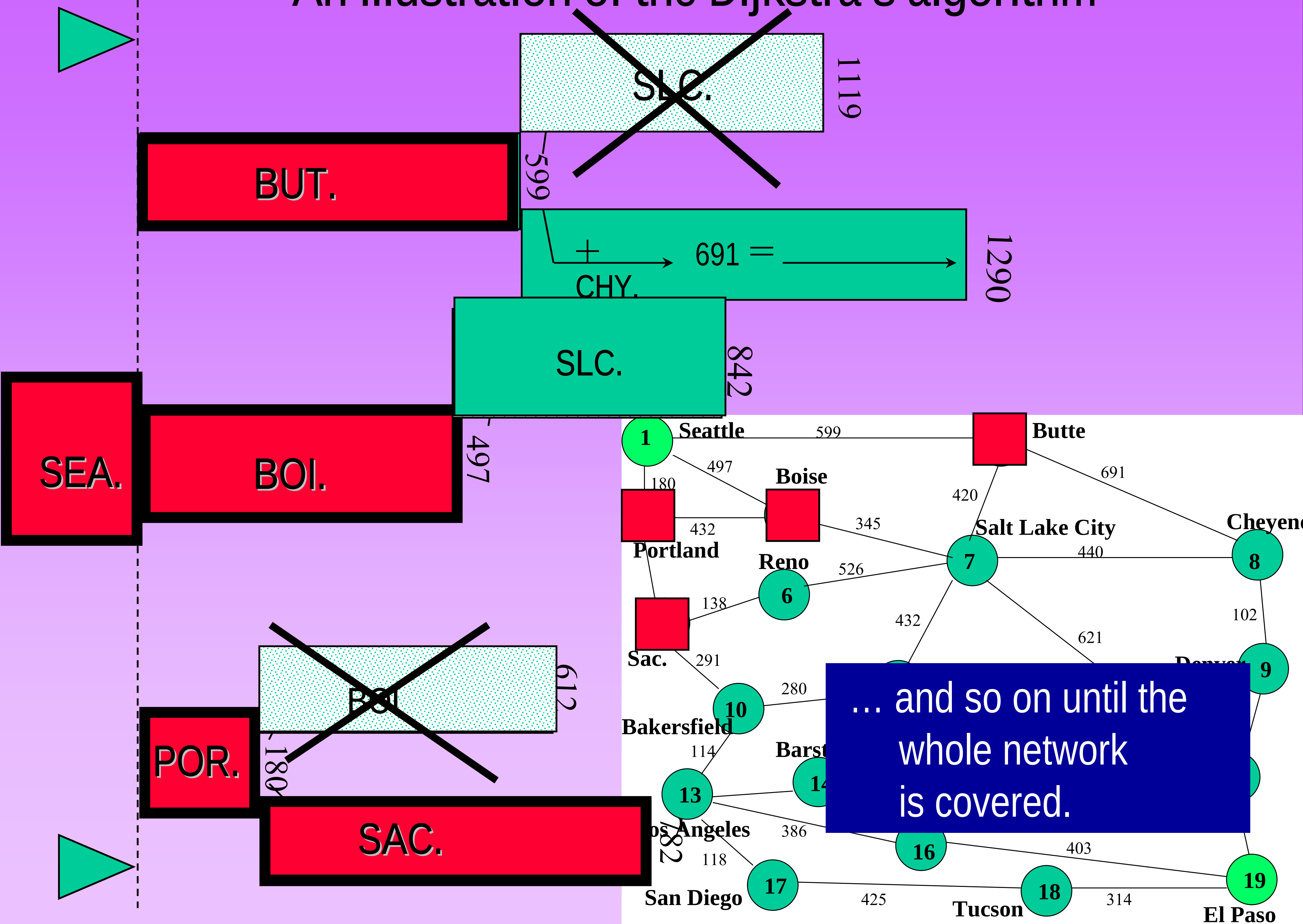
- Solution - a network approach

The Dijkstra’s algorithm:

- Find the shortest distance from the “START” to each other node, in the order of the closet nodes to the “START”.
- Once the shortest route to the m closest node is determined, the (m+1) closest can be easily determined.
- This algorithm finds the shortest route from the start to all the nodes in the network.

An illustration of the Dijkstra's algorithm



6.6 The Minimal Spanning Tree

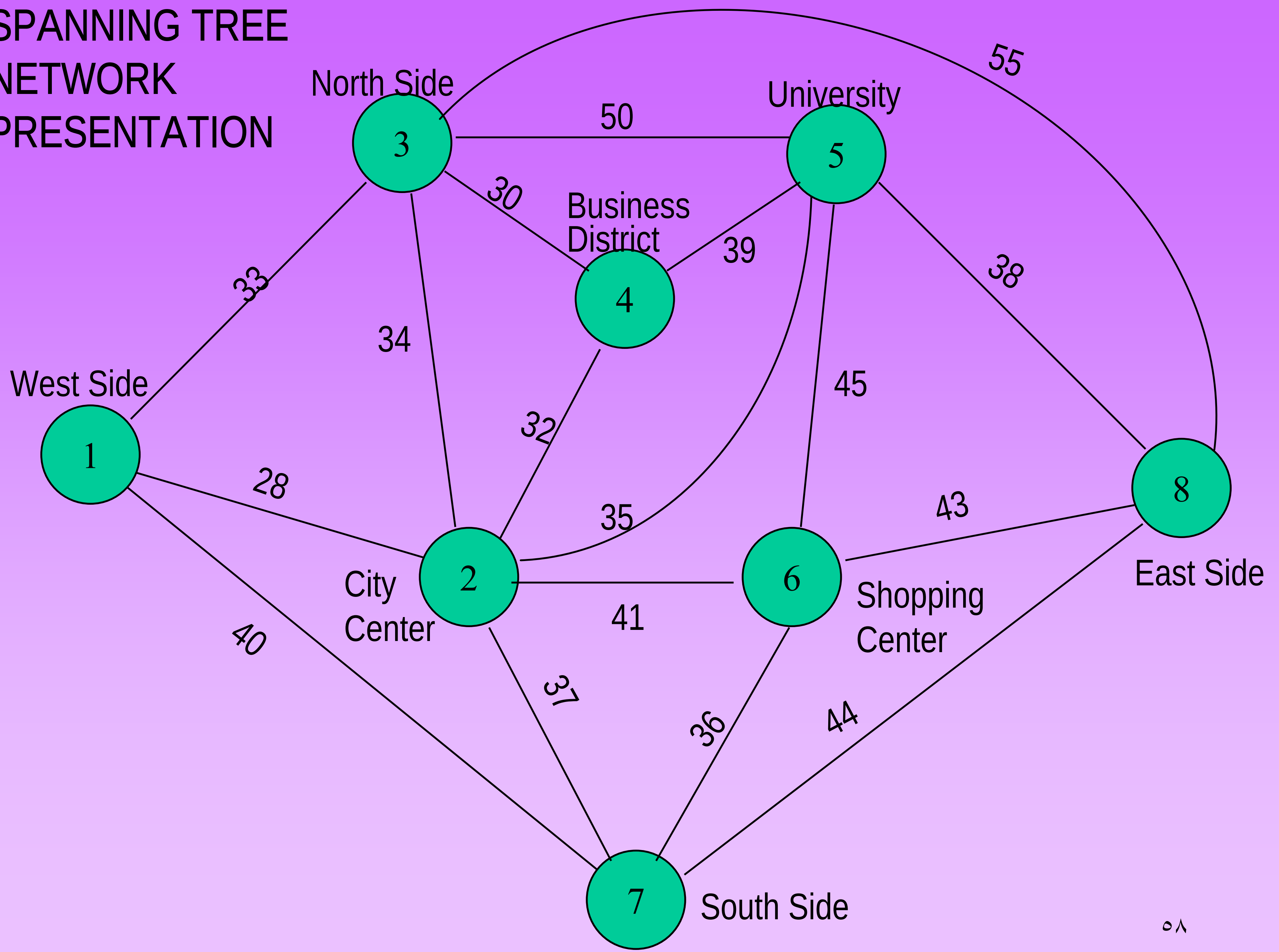
- This problem arises when all the nodes of a given network must be connected to one another, without any loop.

- The minimal spanning tree approach is appropriate for problems for which redundancy is expensive, or the flow along the arcs is considered instantaneous.

THE METROPOLITAN TRANSIT DISTRICT

- The City of Vancouver is planning the development of a new light rail transportation system.
- The system should link 8 residential and commercial centers.
- The Metropolitan transit district needs to select the set of lines that will connect all the centers at a minimum total cost.
- The network describes:
  - feasible lines that have been drafted,
  - minimum possible cost for taxpayers per line.

SPANNING TREE NETWORK PRESENTATION



- Solution - a network approach
  - The algorithm that solves this problem is a very easy ("trivial") procedure.
  - It belongs to a class of "greedy" algorithms.
  - The algorithm:
    - Start by selecting the arc with the smallest arc length.
    - At each iteration, add the next smallest arc length to the set of arcs already selected (provided no loop is constructed).
    - Finish when all nodes are connected.

- Computer solution
  - Input consists of the number of nodes, the arc length, and the network description.

Network Modeling

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Solution for Minimal Spanning Tree Problem METROPOLITAN TRANSIT DISTRICT

	From Node	Connect To	Distance/Cost		From Node	Connect To	Distance/Cost
1	WEST SD	CITY CEN	28	5	SOUTH SD	SHOPPING	36
2	BUSINESS	NORTH SD	30	6	CITY CEN	SOUTH SD	37
3	CITY CEN	BUSINESS	32	7	UNIVERS	EAST SD	38
4	CITY CEN	UNIVERS	35				
	Total	Minimal	Connected	Distance	or Cost	=	236

WINQSB Optimal Solution

Results Run and Solve the Problem

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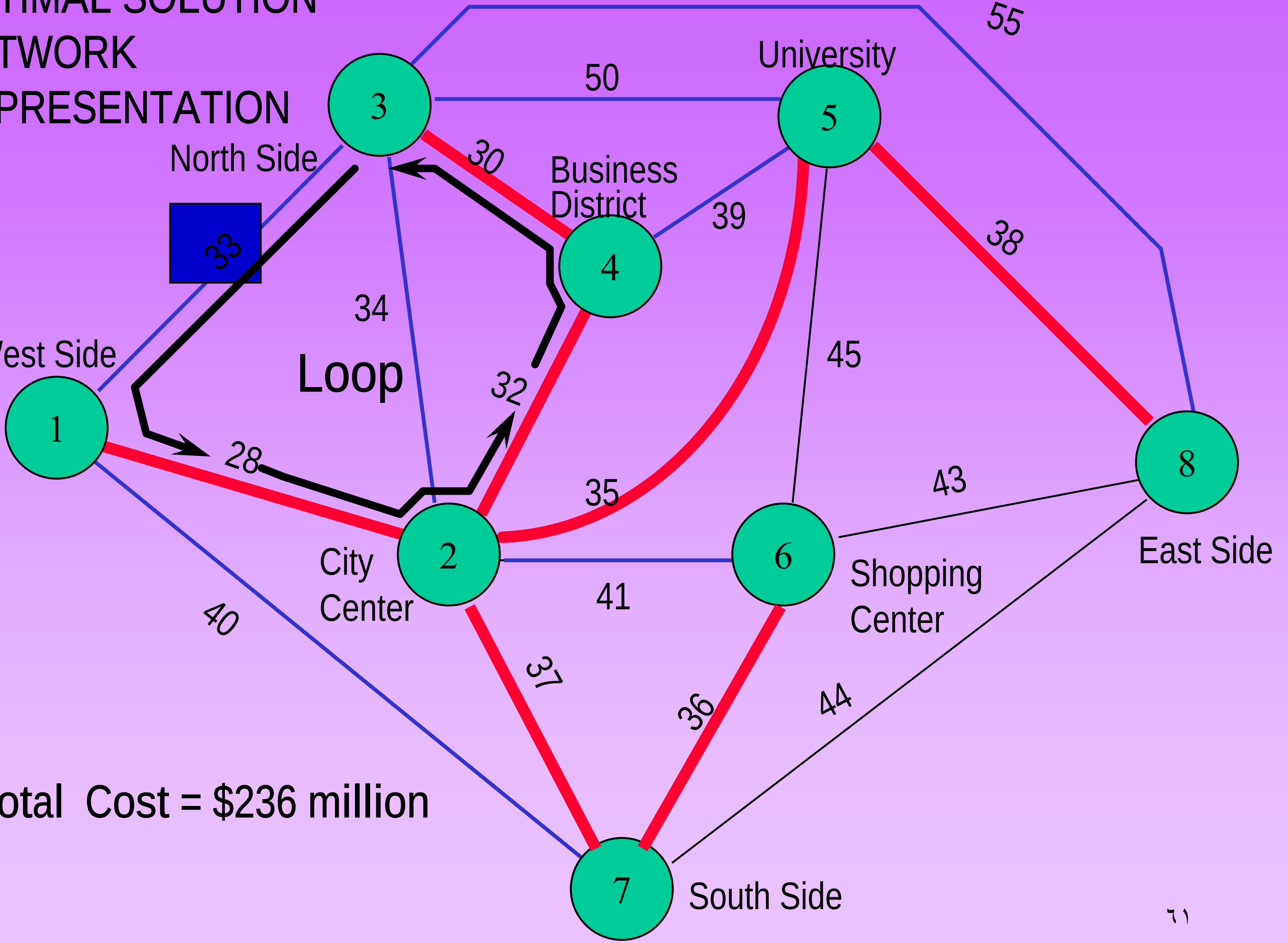
OPTIMAL SOLUTION NETWORK REPRESENTATION



Total Cost = $236 million

6.7 The Maximal Flow Problem

- The model is designed to reduce or eliminate bottlenecks between a certain starting point and some destination of a given network.
- A flow travels from a single source to a single sink over arcs connecting intermediate nodes.
- Each arc has a capacity that cannot be exceeded.
- Capacities need not be the same in each direction

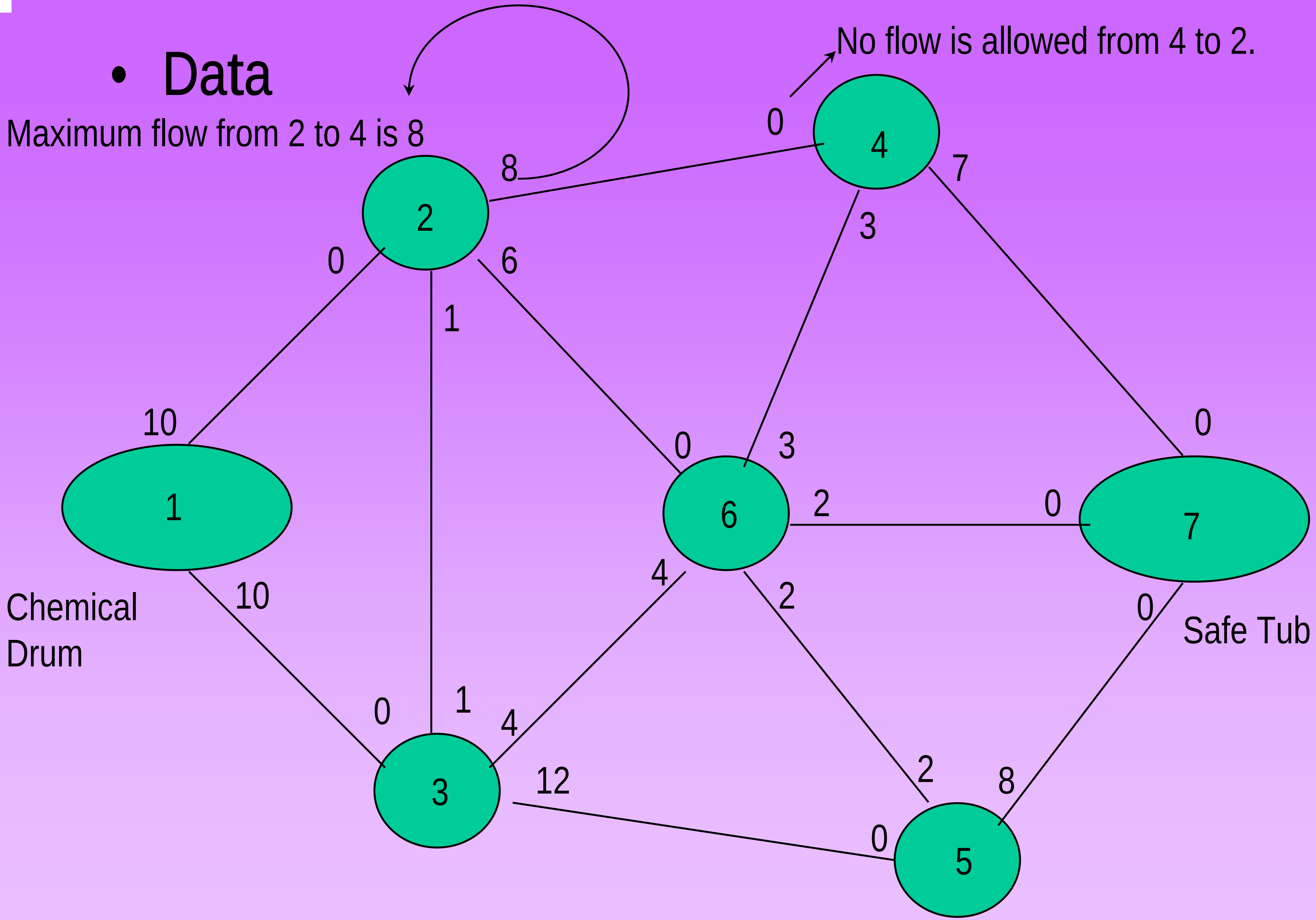
- Problem definition
  - There is a source node (labeled 1), from which the network flow emanates.
  - There is a terminal node (labeled n), into which all network flow is eventually deposited.
  - There are n - 2 intermediate nodes (labeled 2, 3,…,n-1), where the node inflow is equal to the node outflow.
  - There are capacities C_{ij} for flow on the arc from node i to node j, and capacities C_{ji} for the opposite direction.

The objective is to find the maximum total flow out of node 1 that can flow into node n without exceeding the capacities on the arcs.

UNITED CHEMICAL COMPANY

- United Chemical produces pesticides and lawn care products.
- Poisonous chemicals needed for the production process are held in a huge drum.
- A network of pipes and valves regulates the chemical flow from the drum to different production areas.
- The safety division must plan a procedure to empty the drum as fast as possible into a safety tub in the disposal area, using the same network of pipes and valves.
- The plan must determine
  – which valves to open and shut

- Data



Maximum flow from 2 to 4 is 8
No flow is allowed from 4 to 2.
Chemical
Drum
Safe Tub
1
2
3
4
5
6
7
8
0
0
7
6
1
3
10
10
0
3
2
0
0
4
2
0
1
0
4
12
2
8
0

- Solution - linear programming approach
  - Decision variables

    X_{ij} - the flow from node i to node j on the arc that connects these two nodes

  - Objective function - Maximize the flow out of node 1

    Max X12 + X13

  - Constraints
    - [Total flow Out of node 1] = [Total flow entering node 7]

      X12 +X13 = X47 + X57 + X67

    - [For each intermediate node: Flow into = flow out from]

Node 2: X12 + X32 = X23 +X24 + X26

Node 3: X13 +X23 + 63 = X32 +X35 + X36

Node 4: X24 +X64 = X46 + X47

Node 5: X35 +X65 = X56 + X57

Node 6: X26 +X36 + X46 +X56 = X63 +X64 +X65 + X67

- Flow cannot exceed arc capacities

$X12 \le 10$; $X13 \le 10$; $X23 \le 1$; $X24 \le 8$; $X26 \le 6$; $X32 \le 1$;

$X35 \le 15$; $X36 \le 4$; $X46 \le 3$; $X47 \le 7$; $X56 \le 2$; $X57 \le 8$;

$X63 \le 4$; $X64 \le 3$; $X65 \le 2$; $X67 \le 2$;

- Flow cannot be negative: All $X_{ij} \ge 0$

- This problem is relatively small and a solution can be obtained rather quickly by a linear programming model.

- However, for large network problems, there is a more efficient approach

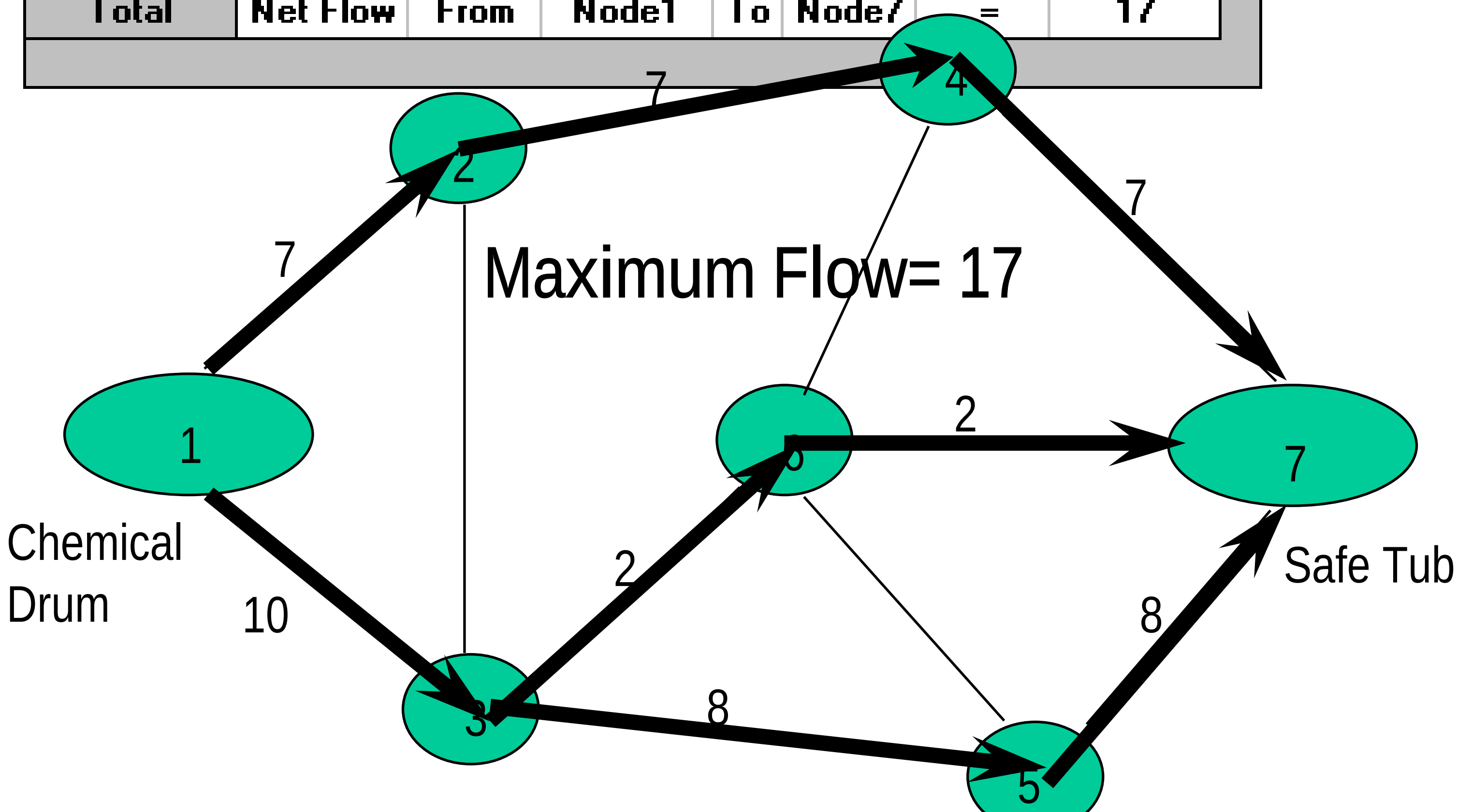
- Solution - the network approach
  - The basic idea is as follows:
    - Find a path with unused capacity on each of its arcs.
    - Augment the flow on these arcs by the minimum remaining capacity of any arc on the path.
    - Repeat this procedure until no path from the source to the sink can be found in which all arcs have residual positive capacity.
    - Computer solution
  - Designate a source node and a sink node.
  - Define the capacities along the arcs in the network. (Allow for different forward and backward capacities.)
  - A WINQSB solution is shown next

Network Modeling - [Solution for Maximal Flow Problem UNITED CHEMICAL]

File

The WINQSB Maximum Flow Optimal Solution

07-10-1997	From	To	Net Flow		From	To	Net Flow
1	Node1	Node2	7	5	Node3	Node6	2
2	Node1	Node3	10	6	Node4	Node7	7
3	Node2	Node4	7	7	Node5	Node7	8
4	Node3	Node5	8	8	Node6	Node7	2
Total	Net Flow	From	Node1	To	Node7	=	17



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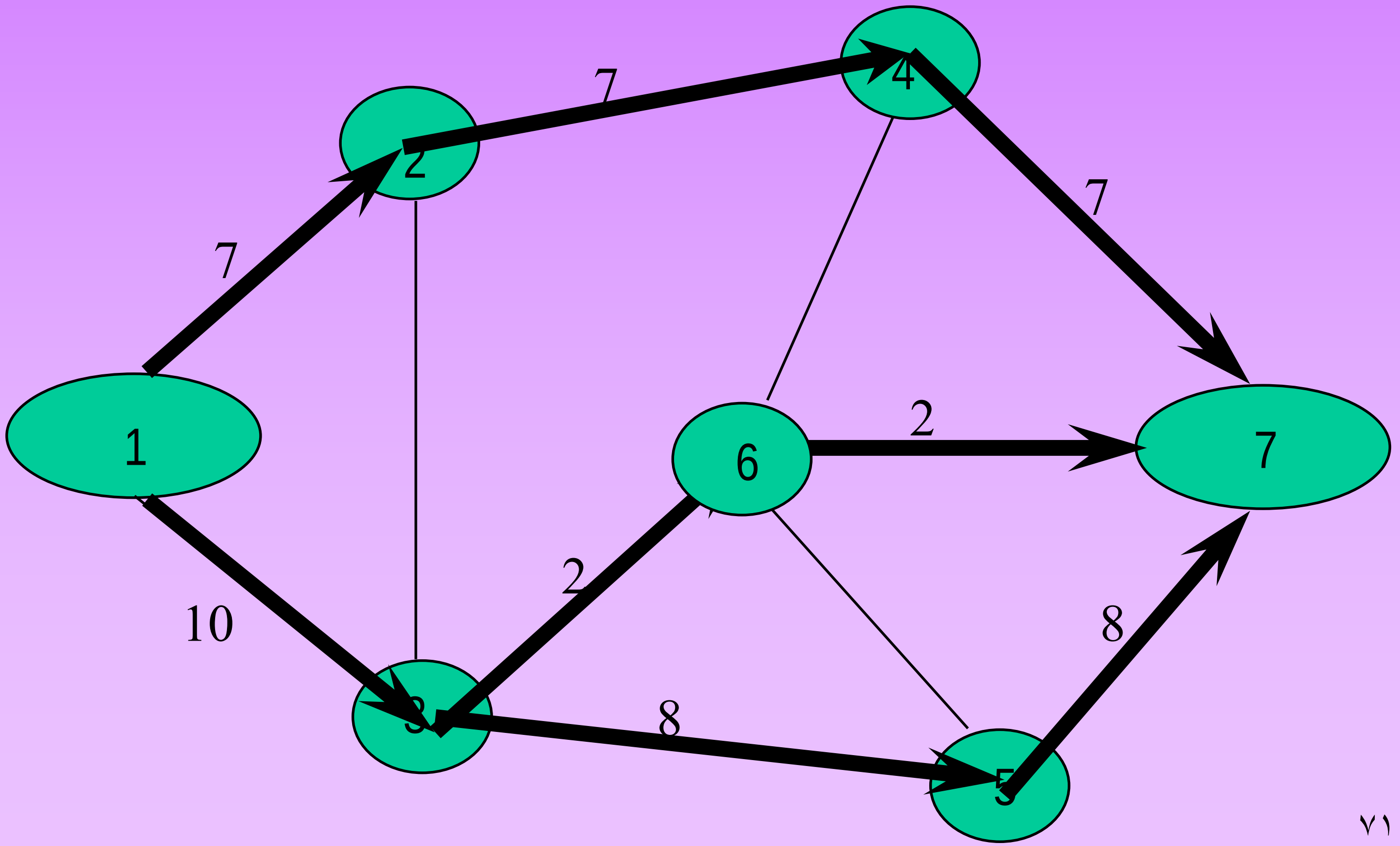
- The role of cuts in a Maximum Flow network
  - The value of the maximum flow = the sum of the capacities of the minimum cut.
  - All arcs on the minimum cut are saturated by the maximum flow.



- Special cases
  - More than one sources node and/or more than one sink node.
  - Add one “supersource” and/or one “supersink”.

Supersource capacity = Total flow capacity out of each source.

Supersink capacity = Total capacities into each sink.